Int. J. Math. And Appl., 9(3)(2021), 1–18
ISSN: 2347-1557
Available Online: http://ijmaa.in/

International Journal of *Mathematics And its Applications*

A Solution for the Non-Cooperative Equilibrium Problem of Three Person via Fixed Point Theory

Soniya Patel[1,*] and R. S. Patel[1]

1 Department of Mathematics, Government Autonomous P.G. College, Satna, Madhya Pradesh, India.

Abstract: In this paper, we investigate the non-cooperative equilibrium problem of three person games in the setting of game theory and proposed a solution via couple fixed point results in the context of partial metric spaces. We also realized the our Tripled fixed point results can be applied to get a solution of a class of nonlinear Fredholm type integral equations.

MSC: 46J10, 46J15, 47H10.

Keywords: Tripled fixed point, partial metric, F-contractions, non-cooperative equilibrium.

1. Introduction

It is very well known fact that real world problem can be modeled as a mathematical equation. For the existence of a solution of such problems has been investigated by Several branches of mathematics, such as, differential equations, integral equations, functional equations, partial differential equations, random differential equations, etc. have proposed solutions for such problems via fixed point theory. But, application area of the fixed point theory is not only limited by Mathematics, but also other quantitative sciences, such as, computer science, economics, biology, physics etc. Game theory, branch of economics, has used fixed point theory techniques and approaches to solve its own problems.

Game theory can be regarded as a formal (mathematical) way to study games. Indeed, we consider the games as conflicts where some number of individuals (called players) take part and each one tries to maximize his utility in taking part in the conflict. Games can be classified in many ways, but here we focus on the following classification: Cooperative games, in which, players are allowed to corporate and non-cooperative games,in which, players are not allowed to corporate. In the sequel, we shall demonstrate how the question of existence of equilibria is related to the question of the existence of a fixed point.

Throughout the paper, we follow the notion and notation in [24]. We recall some basic concepts.

A *three person game* $\mathcal{G}$ in normal form consists of the following data:

(1). topological spaces $\mathcal{S}_1$, $\mathcal{S}_2$ and $\mathcal{S}_3$, the so called strategies for player 1 resp. player 2 and player 3,

(2). a topological subspace $U \subset \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3$ of allowed strategy pair,

(3). a triloss operator

* *E-mail: patelrajesh20081975@gmail.com (Research Scholar)*

$$\begin{aligned} L &: U \to \mathbb{R}^3 \\ (s_1,s_2,s_3) &\mapsto (L_1(s_1,s_2,s_3); L_2(s_1,s_2,s_3); L_3(s_1,s_2,s_3)), \end{aligned} \tag{1}$$

where, $L_i(s_1,s_2,s_3)$ is the loss of player i if the strategies s_1, s_2 and s_3 are played.
A pair $(\overline{s}_1,\overline{s}_2,\overline{s}_3) \in U$ is called a *non-cooperative equilibrium* if

$$\begin{aligned} L_1(\overline{s}_1,\overline{s}_2,\overline{s}_3) &\le L_1(s_1,\overline{s}_2,\overline{s}_3),\ \forall s_1 \in S_1 \\ L_2(\overline{s}_1,\overline{s}_2,\overline{s}_3) &\le L_2(\overline{s}_1,s_2,\overline{s}_3),\ \forall s_2 \in S_2 \\ L_3(\overline{s}_1,\overline{s}_2,\overline{s}_3) &\le L_2(\overline{s}_1,\overline{s}_2,s_3),\ \forall s_3 \in S_3. \end{aligned} \tag{2}$$

Assume that there exist mappings

$$\begin{aligned} C_1 &: \mathcal{S}_3 \to \mathcal{S}_1 \\ C_2 &: \mathcal{S}_2 \to \mathcal{S}_3 \\ C_3 &: \mathcal{S}_1 \to \mathcal{S}_2 \end{aligned} \tag{3}$$

such that the following equations hold:

$$\begin{aligned} L_1(C_1(s_3),s_2,s_3) &= \min_{s_1\in S_1} L_1(s_1,s_2,s_3),\ \forall s_2 \in S_2,\quad s_3 \in S_3 \\ L_2(s_1,C_3(s_1),s_3) &= \min_{s_2\in S_2} L_2(s_1,s_2,s_3),\ \forall s_1 \in S_1,\quad s_3 \in S_3 \\ L_3(s_1,s_2,C_2(s_3)) &= \min_{s_3\in S_3} L_3(s_1,s_2,s_3),\ \forall s_1 \in S_1,\quad s_2 \in S_2. \end{aligned} \tag{4}$$

Such mappings C_1, C_2 and C_3 are called *optimal decision rules.* Then any solution $(\overline{s}_1, s_2, s_3)$ of the system

$$\begin{aligned} C_1(\overline{s}_3) &= \overline{s}_1 \\ C_2(\overline{s}_2) &= \overline{s}_3 \\ C_3(\overline{s}_1) &= \overline{s}_2 \end{aligned} \tag{5}$$

is a non-cooperative equilibrium. Denoting with F the function

$$\begin{aligned} F &: \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3 \to \mathcal{S}_1 \times \mathcal{S}_2 \times \mathcal{S}_3 \\ (\overline{s}_1,\overline{s}_2,\overline{s}_3) &\mapsto (C_1(\overline{s}_3); C_3(\overline{s}_1); C_2(\overline{s}_2)), \end{aligned} \tag{6}$$

the any tripled fixed point $(\overline{s}_1,\overline{s}_2,\overline{s}_3)$ of F is a non-cooperative equilibrium. Hence, investigation of the existence of a solution for non-cooperative equilibrium is equivalent to search the existence of a tripled fixed point. For more details about game theory can be found [24].
The main goal of the present work is to solve the problem of the non-cooperative equilibrium of three person games. For this purpose, we shall present some tripled fixed point theorems in partial metric spaces. Our aim is to explore not only the results themselves but also their applications to nonlinear integral equations.

2. Preliminaries

The notion partial metric was proposed by Matthews (see [12, 13]) as a generalization of metric concept to get a better results in the branches of computer sciences: Semantics and computer domain. Indeed, partial metric is a function that is obtained from metric by replacing the condition $d(x,x) = 0$ with the condition $d(x,x) \le d(x,y)$ for all x, y. On the last decade, a number of authors have brought into focus on the fixed point problems in the context of partial metric spaces as well as on topological properties of partial metric space see [1, 6, 10, 11] and the related references given therein.

We first need to recall some basic concepts and necessary results. Throughout the paper, $\mathbb{N}$ and $\mathbb{N}_0$ denote the set of positive integers and the set of nonnegative integers, respectively. Similarly, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ represent the set of reals, positive reals and nonnegative reals, respectively.

Definition 2.1 ([6, 12])**.** *Let X be a nonempty set. The mapping $p : X \times X \to [0, \infty)$ is said to be a partial metric on X if for any $x, y, z \in X$ the following conditions hold true:*
(P1) $x = y$ if and only if $p(x, x) = p(y, y) = p(x, y)$.
(P2) $p(x, x) \leq p(x, y)$.
(P3) $p(x, y) = p(y, x)$.
(P4) $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.
The pair (X, p) is then called a partial metric space (in short PMS).

Let (X, p) be a partial metric space. Then, the functions $d_p, d_m : X \times X \to [0, \infty)$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

and

$$d_m(x, y) = \max\{p(x, y) - p(x, x), p(x, y) - p(y, y)\}$$

are (usual) metrics on X. It is easy to check that d_p and d_m are equivalent. Note that each partial metric p on X generates a T_0-topology τ_p with a base of the family of open $p-$balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$.

Definition 2.2 ([1, 6])**.** *Let (X, p) be a partial metric space.*

(1). A sequence $\{x_n\}$ in X converges to $x \in X$ if and only if $p(x, x) = \lim\limits_{n\to\infty} p(x_n, x)$.

(2). A sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if $\lim\limits_{n,m\to\infty} p(x_n, x_m)$ exists (and finite).

(3). (X, p) is called to be complete if every Cauchy sequence $\{x_n\}$ in X converges to $x \in X$.

(4). A mapping $f : X \to X$ is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $f(B(x_0, \delta)) \subset B(f(x_0), \varepsilon)$.

Referring to [21], we say that a sequence $\{x_n\}$ in (X, p) is called *0-Cauchy* sequence if $\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Also, we say that (X, p) is *0-complete* if every 0-Cauchy sequence in X converges, with respect to the partial metric p, to a point $x \in X$ such that $p(x, x) = 0$. Notice that if (X, p) is complete, then it is 0-complete, but the converse does not hold. Moreover, every 0-Cauchy sequence in (X, p) is Cauchy in (X, d_p).

Example 2.3 ([12, 21])**.**

(1). Let $X = [0, +\infty)$ and define $p(x, y) = \max\{x, y\}$, for all $x, y \in X$. Then (X, p) is a complete partial metric space. It is clear that p is not a (usual) metric.

(2). Let $X = [0, +\infty) \cap \mathbb{Q}$, where $\mathbb{Q}$ is the set of rational numbers.

Define $p(x, y) = \max\{x, y\}$, for all $x, y \in X$. Then (X, p) is a 0-complete partial metric space which is not complete.

Proposition 2.4 ([1, 6]). *Let (X, p) be a partial metric space.*

(1). A sequence $\{x_n\}$ is a Cauchy sequence in (X, p) if and only if $\{x_n\}$ is a Cauchy sequence in (X, d_p).

(2). (X, p) is complete if and only if (X, d_p) complete. Moreover,

$$\lim_{n\to\infty} d_p(x_n, x) = 0 \Leftrightarrow \lim_{n\to\infty} p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_m, x_n).$$

The following lemmas have an important role in the proof of theorems.

Lemma 2.5 ([1, 11]). *Assume $x_n \to z$ as $n \to \infty$ in a PMS (X, p) such that $p(z, z) = 0$. Then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for every $y \in X$.*

Lemma 2.6 ([1, 11]). *Let (X, p) be a complete PMS. Then*

(1). If $p(x, y) = 0$ then $x = y$.

(2). If $x \neq y$, then $p(x, y) > 0$.

Lemma 2.7 ([1, 11]). *Let (X, p) be a PMS. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$ for all $x_n, y_n, x, y \in X$ then $p(x_n, y_n) \to p(x, y)$ as $n \to \infty$.*

The existence and uniqueness of fixed points fixed points of contractive type mappings in partially ordered metric spaces has been considered recently by several authors: Ran and Reurings [27], Nieto and Rodriguez-Lopez [28, 29]. Following this initial result, Bhaskar and Lakshmikantham [18] proposed the notion of mixed monotone property and get Tripled fixed point results in the setting of partially ordered metric spaces (see also, [19, 30, 31] and the related references therein.) Later, it was reported that the most of the Tripled fixed point results can be derived from the existence results, and vice versa, see e.g. [32–34]. On the other hand, Tripled fixed point results still have worths regarding their applications. Most of the times, using Tripled fixed point theory is most economical way to solve problem (regarding time and speed of the process.) This paper can be considered as an example in this direction.
Recall that a pair $(x, y, z) \in X \times X \times X$ is called a *tripled fixed point* of the mapping $T : X \times X \times X \to X$ if $T(x, y, z) = x, T(y, x, y) = y, T(z, y, x) = z$.

Definition 2.8. *Let $(X, \leq)$ be a partially ordered set and $T : X \times X \times X \to X$. The mapping T is said to have the mixed monotone property if $T(x, y, z)$ is monotone non-decreasing in x and z and monotone non-increasing in y, that is, for any $x, y, z \in X$*

$$x_1, x_2 \in X, x_1 \leq x_2 \Rightarrow T(x_1, y, z) \leq T(x_2, y, z),$$

$$y_1, y_2 \in X, y_1 \leq y_2 \Rightarrow T(x, y_1, z) \geq T(x, y_2, z),$$

$$z_1, z_2 \in X, z_1 \leq z_2 \Rightarrow T(x, y, z_1) \leq T(x, y, z_2).$$

Next, we introduce a class of functions which plays a crucial role in this paper.
Let $F : \mathbb{R}_0^+ \to \mathbb{R}$ be a mapping satisfying

(F_1) F is strictly increasing and continuous.

(F_2) For each sequence $(a_n) \subset \mathbb{R}_0^+$, $\lim_{n\to\infty} a_n = 0$ if and only if $\lim_{n\to\infty} F(a_n) = -\infty$.

We denote with $\mathcal{F}$ the family of all functions F that satisfy the conditions $(F_1)-(F_2)$ (see [22]). It is easy to check that $F(x)=\ln x$ and $G(x)=\ln x+x$ for all $x\in\mathbb{R}_0^+$ belong to $\mathcal{F}$.

In [20], D. Wardowski introduced the new concept of F - contraction and proved fixed point theorems in the classical setting of metric spaces. In [22], the authors introduced the concept of F - contraction, generalized F - contraction and proved some fixed point theorems for multi-valued mappings in the partial metric spaces (see also, [2, 3]).

Definition 2.9 ([22]). *Let (X,p) be a partial metric space. A mapping $T:X\times X\to X$ is called an F-contraction if there exist $F\in\mathcal{F}$ and $\tau\in\mathbb{R}_0^+$ such that*

$$\tau+F(p(Tx,Ty))\le F(p(x,y))$$

for all $x,y\in X$.

3. Auxiliary Results: Tripled Fixed Points in Partial Metric Spaces

In this section we state and prove some new Tripled fixed point results for F-contractive mappings in the context of complete partial metric spaces.

Theorem 3.1. *Let $(X,\le)$ be a partially ordered set and suppose there exists a partial metric p on X such that (X,p) is a 0-complete partial metric space. Let $T:X\times X\times X\to X$ be a continuous mapping having the mixed monotone property on X. Suppose also that*

(1).

$$\tau+F\big(p(T(x,y,z),T(u,v,w))\big)\le F\Big(\max\big\{p(x,u),p(y,v),p(z,w)\big\}\Big) \tag{7}$$

for all $x\le u, y\ge v, z\le w$, for some $F\in\mathcal{F}$ and $\tau>0$.

(2). There are $x_0,y_0,z_0\in X$ such that $x_0\le T(x_0,y_0,z_0), y_0\ge T(y_0,x_0,y_0), z_0\le T(z_0,y_0,x_0)$.

Then T has a tripled fixed point, that is, there exist $x,y,z\in X$ such that $x=T(x,y,z), y=T(y,x,y), z=T(z,y,x)$.

Proof. Let $x_0,y_0,z_0\in X$ be such that $x_0\le T(x_0,y_0,z_0), y_0\ge T(y_0,x_0,y_0), z_0\le T(z_0,y_0,x_0)$. Let $x_1=T(x_0,y_0,z_0)$, $y_1=T(y_0,x_0,y_0)$ and $z_1=T(z_0,y_0,x_0)$. Then $x_0\le x_1$, $y_0\ge y_1$ and $z_0\le z_1$. Again, let $x_2=T(x_1,y_1,z_1)$, $y_2=T(y_1,x_1,y_1)$ and $z_2=T(z_1,y_1,x_1)$. Since T has the mixed monotone property, the we have $x_1\le x_2$, $y_1\ge y_2$ and $z_1\le z_2$. Continuing this way, we get three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that $x_{n+1}=T(x_n,y_n,z_n), y_{n+1}=T(y_n,x_n,y_n), z_{n+1}=T(z_n,y_n,x_n)$ and

$$\begin{aligned}
&x_0\le x_1\le x_2\le ...\le x_n\le x_{n+1}...,\\
&y_0\ge y_1\ge y_2\ge ...\ge y_n\ge y_{n+1}...,\\
&z_0\le z_1\le z_2\le ...\le z_n\le z_{n+1}...
\end{aligned}$$

Now, for each $n=0,1,2,...$, we have

$$\begin{aligned}
\tau+F\big(p(x_n,x_{n+1})\big)&=\tau+F\Big(p\big(T(x_{n-1},y_{n-1},z_{n-1}),T(x_n,y_n,z_n)\big)\Big)\\
&\le F\Big(\max\{(p(x_{n-1},x_n)),p(y_{n-1},y_n),p(z_{n-1},z_n)\}\Big) \qquad (8)\\
\tau+F\big(p(y_n,y_{n+1})\big)&=\tau+F\Big(p\big(T(y_{n-1},x_{n-1},y_{n-1}),T(y_n,x_n,(y_n))\Big)
\end{aligned}$$

$$\leq F\Big(\max\{p(y_{n-1},y_n),p(x_{n-1},x_n),p(y_{n-1},y_n)\}\Big) \tag{9}$$

$$\tau+F\big(p(z_n,z_{n+1})\big)=\tau+F\Big(p\big(T(z_{n-1},y_{n-1},x_{n-1}),T(z_n,y_n,x_n)\big)\Big)$$
$$\leq F\Big(\max\{(p(z_{n-1},z_n)),p(y_{n-1},y_n),p(x_{n-1},x_n)\}\Big) \tag{10}$$

Since (8), (9), (10) and F is increasing we get that

$$\tau+F\Big(\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1}),p(z_n,z_{n+1})\}\Big)\leq F\Big(\max\{p(x_{n-1},x_n),p(y_{n-1},y_n),p(z_{n-1},z_n)\}\Big). \tag{11}$$

It follows that

$$\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1}),p(z_n,z_{n+1})\}\leq \max\{p(x_{n-1},x_n),p(y_{n-1},y_n),p(z_{n-1},z_n)\}$$

for all $n=1,2,...$ Hence, the sequence $r_n:=\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1}),p(z_n,z_{n+1})\}$ is a non-increasing. Thus, there is $r\geq 0$ such that $\lim_{n\to\infty} r_n=r$. Since F is continuous, letting $n\to\infty$ in (11), we arrive at

$$\tau+F(r)\leq F(r).$$

Since $\tau>0$ and the definition of F, we can deduce that $F(r)=-\infty$, this implies that $r=0$. Therefore

$$\lim_{n\to\infty}\max\{p(x_n,x_{n+1}),p(y_n,y_{n+1}),p(z_n,z_{n+1})\}=0. \tag{12}$$

Next, we shall claim that

$$\lim_{m,n\to\infty}\max\{p(x_m,x_n),p(y_n,y_m),p(z_n,z_m)\}=0. \tag{13}$$

Suppose to the contrary, then there exits $\varepsilon>0$ for that we can seek three subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of respectively $\{x_m\}$ and $\{x_n\}$ such that $n(k)$ is the smallest index for which

$$n(k)>m(k)>k,\ \ \max\{p(x_{m(k)},x_{n(k)}),p(y_{n(k)},y_{m(k)}),p(z_{n(k)},z_{m(k)})\}\geq\varepsilon. \tag{14}$$

This mean that

$$\max\{p(x_{m(k)},x_{n(k)-1}),p(y_{m(k)},y_{n(k)-1}),p(z_{m(k)},z_{n(k)-1})\}<\varepsilon, \tag{15}$$

and we obtain

$$\begin{aligned}p(x_{m(k)},x_{n(k)})&\leq p(x_{m(k)},x_{n(k)-1})+p(x_{n(k)-1},x_{n(k)})-p(x_{n(k)-1},x_{n(k)-1})\\&\leq p(x_{m(k)},x_{n(k)-1})+p(x_{n(k)-1},x_{n(k)})<\varepsilon+p(x_{n(k)-1},x_{n(k)}).\end{aligned} \tag{16}$$

Similarly, we get that

$$p(y_{m(k)},y_{n(k)})<\varepsilon+p(y_{n(k)-1},y_{n(k)}). \tag{17}$$

and

$$p(z_{m(k)},z_{n(k)})<\varepsilon+p(z_{n(k)-1},z_{n(k)}). \tag{18}$$

Combining (14), (16), (17) and (18), we obtain

$$\begin{aligned}\varepsilon&\leq\max\{p(x_{m(k)},x_{n(k)}),p(y_{m(k)},y_{n(k)}),p(z_{m(k)},z_{n(k)})\}\\&\leq\varepsilon+\max\{p(x_{n(k)-1},x_{n(k)}),p(y_{n(k)-1},y_{n(k)}),p(z_{n(k)-1},z_{n(k)})\}\end{aligned} \tag{19}$$

Letting $k \to \infty$ in (19) and using (12), we have

$$\lim_{k\to\infty} \max\{p(x_{m(k)}, x_{n(k)}, p(y_{m(k)}, y_{n(k)}), p(z_{m(k)}, z_{n(k)})\} = \varepsilon. \tag{20}$$

Now, by the fact that

$$\begin{aligned} p(x_{m(k)}, x_{n(k)}) &\le p(x_{m(k)}, x_{n(k)-1}) + p(x_{n(k)-1}, x_{n(k)}) \\ p(y_{m(k)}, y_{n(k)}) &\le p(y_{m(k)}, y_{n(k)-1}) + p(y_{n(k)-1}, y_{n(k)}) \\ p(z_{m(k)}, z_{n(k)}) &\le p(z_{m(k)}, z_{n(k)-1}) + p(z_{n(k)-1}, z_{n(k)}) \end{aligned}$$

we obtain

$$\begin{aligned} \max\{p(x_{m(k)}, x_{n(k)}), p(y_{m(k)}, y_{n(k)}), p(z_{m(k)}, z_{n(k)})\} &\le \max\{p(x_{m(k)}, x_{n(k)-1}), p(y_{m(k)}, y_{n(k)-1}), p(z_{m(k)}, z_{n(k)-1})\} \\ &+ \max\{p(x_{n(k)-1}, x_{n(k)}), p(y_{n(k)-1}, y_{n(k)}), p(z_{n(k)-1}, z_{n(k)})\}. \end{aligned} \tag{21}$$

By the same argument, we also have

$$\begin{aligned} \max\{p(x_{m(k)}, x_{n(k)-1}), p(y_{m(k)}, y_{n(k)-1}), p(z_{m(k)}, z_{n(k)-1})\} &\le \max\{p(x_{m(k)}, x_{n(k)}), p(y_{m(k)}, y_{n(k)}), p(z_{m(k)}, z_{n(k)})\} \\ &+ \max\{p(x_{n(k)-1}, x_{n(k)}), p(y_{n(k)-1}, y_{n(k)}), p(z_{n(k)-1}, z_{n(k)})\}. \end{aligned} \tag{22}$$

Letting $k \to \infty$ in (21), (22) and using (12), (20), we have

$$\lim_{k\to\infty} \max\{p(x_{m(k)}, x_{n(k)-1}, p(y_{m(k)}, y_{n(k)-1}), p(z_{m(k)}, z_{n(k)-1})\} = \varepsilon. \tag{23}$$

Next, since $x_{m(k)} \le x_{n(k)-1}$, $y_{m(k)} \ge y_{n(k)-1}$ and $z_{m(k)} \le z_{n(k)-1}$, we have

$$\begin{aligned} \tau + F\big(p(x_{m(k)+1}, x_{n(k)})\big) &= \tau + F\Big(p\big(T(x_{m(k)}, y_{m(k)}, z_{m(k)}), T(x_{n(k)-1}, y_{n(k)-1}, z_{n(k)-1})\big)\Big) \\ &\le F\Big(\max\{p(x_{m(k)}, x_{n(k)-1}), p(y_{n(k)-1}, y_{m(k)}), p(z_{n(k)-1}, z_{m(k)})\}\Big). \end{aligned}$$

By the same reason, we also have

$$\begin{aligned} \tau + F\big(p(y_{n(k)}, y_{m(k)+1})\big) &= \tau + F\Big(p\big(T(y_{m(k)}, x_{m(k)}, y_{m(k)}), T(y_{n(k)-1}, x_{n(k)-1}, y_{n(k)-1})\big)\Big) \\ &\le F\Big(\max\{p(y_{n(k)-1}, y_{m(k)}), p(x_{n(k)-1}, x_{m(k)}), p(y_{n(k)-1}, y_{m(k)})\}\Big). \end{aligned}$$

and

$$\begin{aligned} \tau + F\big(p(z_{m(k)+1}, z_{n(k)})\big) &= \tau + F\Big(p\big(T(z_{m(k)}, y_{m(k)}, x_{m(k)}), T(z_{n(k)-1}, y_{n(k)-1}, x_{n(k)-1})\big)\Big) \\ &\le F\Big(\max\{p(z_{m(k)}, z_{n(k)-1}), p(y_{n(k)-1}, y_{m(k)}), p(x_{n(k)-1}, x_{m(k)})\}\Big). \end{aligned}$$

Therefore

$$\begin{aligned} \tau + \max\{F\big(p(x_{m(k)+1}, x_{n(k)})\big), F\big(p(y_{n(k)}, y_{m(k)+1})\big), F\big(p(z_{n(k)}, z_{m(k)+1})\big)\} \\ \le F\Big(\max\{p(x_{n(k)-1}, x_{m(k)}), p(y_{n(k)-1}, y_{m(k)}), p(z_{n(k)-1}, z_{m(k)})\}\Big). \end{aligned}$$

Letting $k \to \infty$ and using (23), we arrive at

$$\tau + F(\varepsilon) \le F(\varepsilon).$$

This yields $\varepsilon = 0$, this is a contradiction. Hence, we have proved that

$$\lim_{m,n\to\infty} \max\{p(x_m, x_n), p(y_n, y_m), p(z_n, z_m)\} = 0.$$

This implies that

$$\lim_{m,n\to\infty} p(x_m, x_n) = 0, \quad \lim_{m,n\to\infty} p(y_m, y_n) = 0 \quad \text{and} \quad \lim_{m,n\to\infty} p(z_m, z_n) = 0. \tag{24}$$

Since (X, p) is 0-complete partial metric space, we can find $u, v, w \in X$ such that

$$\lim_{n\to\infty} p(u, x_n) = p(u, u) = 0$$
$$\lim_{n\to\infty} p(v, y_n) = p(v, v) = 0$$
$$\lim_{n\to\infty} p(w, z_n) = p(w, w) = 0.$$

Now, we show that $u = T(u, v, w)$, $v = T(v, u, v)$ and $w = T(w, v, u)$. Indeed, since $u \leq u$, $v \geq v$ and $w \leq w$, we have

$$\tau + F\big(p(T(u, v, w), T(u, v, w))\big) \leq F\Big(\max\{p(u, u), p(v, v), p(w, w)\}\Big) = F(0) = -\infty.$$

This implies that $p(T(u, v, w), T(u, v, w)) = 0$. Since $x_n \to u$, $y_n \to v$, $z_n \to w$ as $n \to \infty$ in (X, p) and T is continuous, we have $T(x_n, y_n, z_n) \to T(u, v, w)$ in (X, p), this mean that

$$\lim_{n\to\infty} p(x_{n+1}, T(u, v, w)) = \lim_{n\to\infty} p(T(x_n, y_n, z_n), T(u, v, w)) = 0.$$

Now, we have

$$p(u, T(u, v, w)) \leq p(u, x_{n+1}) + p(x_{n+1}, T(u, v, w)) - p(x_{n+1}, x_{n+1}).$$

Letting $n \to \infty$, we get that $p(u, T(u, v, w)) = 0$, and so $u = T(u, v, w)$. By the same argument, we also have $p(v, T(v, u, v)) = 0$, and so $v = T(v, u, v)$ also $p(w, T(w, v, u)) = 0$, and so $w = T(w, v, u)$. □

In the next theorem, we omit the continuity hypothesis of T.

Theorem 3.2. *Let $(X, \leq)$ be a partially ordered set and suppose there exists a partial metric p on X such that (X, p) is a 0-complete partial metric space. Let $T : X \times X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that:*

(1).

$$\tau + F\big(p(T(x, y, z), T(u, v, w))\big) \leq F\Big(\max\big\{p(x, u), p(y, v), p(z, w)\big\}\Big) \tag{25}$$

for all $x \leq u, y \geq v, z \leq w$, for some $F \in \mathcal{F}$ and $\tau > 0$.

(2). There are $x_0, y_0, z_0 \in X$ such that $x_0 \leq T(x_0, y_0, z_0)$, $y_0 \geq T(y_0, x_0, y_0)$ and $z_0 \leq T(z_0, y_0, x_0)$.
Also, assume that X has the properties:

(i). If a non-decreasing sequence $\{x_n\}$ and $\{z_n\}$ in X converges to x then $x_n \leq x$ and $z_n \leq z$ for all n.

(ii). If a non-increasing sequence $\{y_n\}$ in X converges to y then $y_n \geq y$ for all n.

Then T has a tripled fixed point, that is, there exist $x, y, z \in X$ such that $x = T(x, y, z), y = T(y, x, y), z = T(z, y, x)$.

Proof. Following the line of the proof of Theorem 3.1. Hence, we only need show that

$$\lim_{n\to\infty} p(x_{n+1}, F(u, v, w)) = \lim_{n\to\infty} p(T(x_n, y_n, z_n), T(u, v, w)) = 0.$$

Under conditions (i) and (ii). Indeed, we have $x_n \leq u$, $y_n \geq v$ and $z_n \leq w$ for all n. Applying (7), we have

$$\begin{aligned} \tau + F(p(x_{n+1}, T(u, v, w))) &= \tau + F(p(F(x_n, y_n, z_n), F(u, v, w))) \\ &\leq F(\max\{p(x_n, u), p(y_n, v), p(z_n, w)\}). \end{aligned}$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} F(p(x_{n+1}, T(u, v, w))) = -\infty.$$

Hence

$$\lim_{n\to\infty} p(x_{n+1}, F(u, v, w)) = \lim_{n\to\infty} p(T(x_n, y_n, z_n), T(u, v, w)) = 0.$$

□

We easily get the following corollary.

Corollary 3.3. *Let $(X, \leq)$ be a partially ordered set and suppose there exists a partial metric p on X such that (X, p) is a* 0- *complete partial metric space. Let $T : X \times X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that:*

(1).

$$\tau + F\big(p(T(x, y, z), T(u, v, w))\big) \leq F\Big(\frac{p(x, u)) + p(y, v) + p(z, w)}{3}\Big) \tag{26}$$

for all $x \leq u, y \geq v, z \leq w$, for some $F \in \mathcal{F}$ and $\tau > 0$.

(2). There are $x_0, y_0, z_0 \in X$ such that $x_0 \leq T(x_0, y_0, z_0), y_0 \geq T(y_0, x_0, y_0), z_0 \leq T(z_0, y_0, x_0)$.

Also, assume either

(a). T is continuous; or

(b). X has the properties:

(i). If a non-decreasing sequence $\{x_n\}$ and $\{z_n\}$ in X converges to x and z respectively then $x_n \leq x$ and $z_n \leq z$ for all n.

(ii). If a non-increasing sequence $\{y_n\}$ in X converges to y then $y_n \geq y$ for all n.

Then T has a Tripled fixed point, that is, there exist $x, y \in X$ such that $x = T(x, y), y = T(y, x)$.

Proof. By the fact that

$$\frac{p(x, u) + p(y, v) + p(z, w)}{3} \leq \max\{p(x, u), p(y, v), p(z, w)\}$$

for all $x, y, z, u, v, w \in X$, the condition (26) implies the condition (7). Therefore, the result is desired from Theorem 3.1 and Theorem 3.2. □

The following corollary state that T has a fixed point the under certain condition.

Corollary 3.4. *In addition to the hypotheses of Corollary 3.3, if x_0 and y_0 are comparable then T has a unique fixed point, that is, there exists $x \in X$ such that $T(x,x,x) = x$.*

Proof. Since x_0, y_0, z_0 are comparable, we have $x_0 \geq y_0 \geq z_0$ or $x_0 \leq y_0 \leq z_0$. Suppose we are in the first case. Then, by the mixed monotone property of T, we have

$$x_1 = T(x_0, y_0, z_0) \geq T(y_0, x_0, y_0) = y_1,$$

and, hence, by induction one obtains

$$x_n \geq y_n \geq z_n \text{ for all } n \geq 0.$$

Now, since $x = \lim_{n\to\infty} x_{n+1}, y = \lim_{n\to\infty} y_{n+1}, z = \lim_{n\to\infty} z_{n+1}$ we have $p(x,y) = \lim\limits_{n\to\infty} p(x_{n+1}, y_{n+1})$. On the other hand, we have

$$\begin{aligned} \tau + F\big(p(x_{n+1}, y_{n+1})\big) &= \tau + F\Big(p\big(T(x_n, y_n, z_n), T(y_n, x_n, y_n)\big)\Big) \\ &\leq F\Big(\max\{(p(x_n, x_n)), p(y_n, y_n), p(z_n, z_n)\}\Big). \end{aligned} \tag{27}$$

Following the Lemma 2.7, we also have

$$\lim_{n\to\infty} p(x_n, x_n) = \lim_{n\to\infty} p(y_n, y_n) = \lim_{n\to\infty} p(z_n, z_n) = 0.$$

Letting $n \to \infty$ in (27), we arrive at $\lim\limits_{n\to\infty} p(x_{n+1}, y_{n+1}) = 0$. Therefore $p(x,y) = 0$, or $x = y = z$. Hence $T(x,x,x) = x$. □

Remark 3.5. *We underline the fact that the tripled fixed point theorem in this paper can be observed from the fixed point result of single mapping by using the techniques in [32–34]. On the other hand, we prefer to keep the proofs for the sake of the completeness.*

4. Main Result: Non-cooperative Equilibrium Problem for Three Players

In this section, by using tripled fixed point theorems, we shall show that a three person game has a non-cooperative equilibrium. The reader may consult excellent sources of general concepts of three person games in [24] and [23]. Let (S, p) be a 0-complete partial metric space. Suppose that S has a partially order relation $\leq$. We consider a three person game $\mathcal{G}$ in normal form consists of following data:

(1). $S_1 = S$, $S_2 = S$ and $S_3 = S$ are strategies for player 1 and respectively player 2, player 3;

(2). The set $U = S_1 \times S_2 \times S_3$ of allowed strategies pairs;

(3). The biloss operator

$$\begin{aligned} &L : U \to \mathbb{R}^3 \\ &(s_1, s_2, s_3) \mapsto (L_1(s_1, s_2, s_3); L_2(s_1, s_2, s_3); L_3(s_1, s_2, s_3)), \end{aligned} \tag{28}$$

where $L_i(s_1, s_2)$ is the loss of player i if the strategies s_1, s_2 and s_3 are played.

A pair $(\overline{s}_1, \overline{s}_2, \overline{s_3}) \in U$ is called a non-cooperative equilibrium if

$$\begin{aligned} &L_1(\overline{s}_1, \overline{s}_2, \overline{s}_3) \leq L_1(s_1, \overline{s}_2, \overline{s}_3)), \ \forall s_1 \in S_1 \\ &L_2(\overline{s}_1, \overline{s}_2, \overline{s}_3) \leq L_2(\overline{s}_1, s_2, \overline{s}_3)), \ \forall s_2 \in S_2 \\ &L_3(\overline{s}_1, \overline{s}_2, \overline{s}_3) \leq L_2(\overline{s}_1, \overline{s}_2, s_3)), \ \forall s_3 \in S_3. \end{aligned} \tag{29}$$

This mean that

$$\begin{aligned} L_1(\overline{s}_1,\overline{s}_2,\overline{s}_3) &= \min_{s_1\in S_1} L_1(s_1,\overline{s}_2,\overline{s}_3)) \\ L_2(\overline{s}_1,\overline{s}_2,\overline{s}_3) &= \min_{s_2\in S_2} L_2(\overline{s}_1,s_2,\overline{s}_3)) \\ L_3(\overline{s}_1,\overline{s}_2,\overline{s}_3) &= \min_{s_3\in S_3} L_2(\overline{s}_1,\overline{s}_2,s_3)). \end{aligned} \tag{30}$$

To see what strategy pairs are non- cooperative equilibria, one consider the optimal decision rules C_1, C_2, C_3 such that

$$\begin{aligned} L_1(C_1(s_3),s_2,s_3) &= \min_{s_1\in S_1} L_1(s_1,s_2,s_3) \\ L_2(s_1,C_2(s_2),s_3) &= \min_{s_2\in S_2} L_2(s_1,s_2,s_3) \\ L_3(s_1,s_2,C_3(s_1)) &= \min_{s_3\in S_3} L_3(s_1,s_2,s_3). \end{aligned} \tag{31}$$

Then any fixed point of the map

$$(s_1,s_2,s_3)\mapsto \big(C_1(s_3),C_2(s_2),C_3(s_1)\big)$$

is a non- cooperative equilibrium.

In this section, we shall consider that $C_1(s)=C_2(s)=C_3(s)$ for all $s\in S$. It is easy to see that if $L_1(s_1,s_2,s_3)=L_2(s_1,s_2,s_3)=L_3(s_1,s_2,s_3)$ for all $(s_1,s_2,s_3)\in S_1\times S_2\times S_3$ then $C_1(s)=C_2(s)=C_3(s)$ and it is not difficult to give example that $C_1(s)=C_2(s)=C_3(s)$ in the case $L_1(s_1,s_2,s_3)\neq L_2(s_1,s_2,s_3)\neq L_3(s_1,s_2,s_3)$. Let $T: S_1\times S_2\times S_3\to\mathbb{R}$ be the map defined by

$$T(x,y,z)=C(z)$$

for all $x,y\in S$. Suppose that T has tripled fixed point $(a,b,c)\in\mathbb{R}$. It follows that

$$\begin{aligned} a &= T(a,b,c)=C(c) \\ b &= T(b,a,b)=C(b) \\ c &= T(c,b,a)=C(a). \end{aligned} \tag{32}$$

and (a,b,c) is fixed point of the map $(s_1,s_2,s_3)\mapsto \big(C(s_3),C(s_2),C(s_1)\big)$. Therefore, the existence tripled fixed point of T implies a non- cooperative equilibrium. Hence, we can reduce the process of proving the existence of a non- cooperative equilibrium to giving existence tripled fixed point of T.

Theorem 4.1. *Let S and $\mathcal{G}$ be as the above mention. Suppose that the optimal decision rule is monotone continuous functions C which satisfies*

(1).

$$\tau+F\big(p(C(x),C(y))\big)\leq F(p(x,y)) \tag{33}$$

for all $x,y\in S$ and $y\geq x$, for some $F\in\mathcal{F}$ and $\tau>0$.

(2). There are $x_0,y_0,z_0\in\mathbb{R}_+$ such that $x_0\leq C(y_0), y_0\geq C(z_0)$.

Then, three person game $\mathcal{G}$ has a non- cooperative equilibrium.

Proof. Let $T: S\times S\times S\to\mathbb{S}$ defined by

$$T(x,y,z)=C(z)$$

for all $x, y, z \in S$. Since C is continuous, we have that T is continuous. Since C is monotone, it is easy to check that T have the mixed monotone property on X. For all $x, y, z, u, v, w \in \mathbb{R}_+$, with $x \leq u$, $y \geq v$, $z \leq w$ we have

$$p(T(x, y, z), T(u, v, w)) = p(C(z), C(w)).$$

Therefore, the condition (7) reduces to

$$\tau + F\big(p(C(z), C(w))\big) \leq F\Big(\max\big\{p(x, u), p(y, v), p(z, w)\big\}\Big), \tag{34}$$

for every $x \leq u$, $y \geq v$ and $z \leq w$. Since

$$\max\{p(x, u), p(y, v), p(z, w)\} \geq p(z, w)$$

and F is increasing, we get that the condition (33) implies to (34). Applying Theorem 3.1, we conclude that T has a tripled fixed point. This implies that the three person games $\mathcal{G}$ has a non- cooperative equilibrium. □

Since every metric is partial metric, we immediately obtain the following corollary.

Corollary 4.2. *Let $\mathcal{G}$ be as the above mention. Suppose that (S, d) is a metric space and the optimal decision rule is monotone continuous functions C which satisfies*

(1).

$$\tau + F\big(d(C(x), C(y))\big) \leq F(d(x, y)) \tag{35}$$

for all $x, y \in \mathbb{S}$ and $x < y$, for some $F \in \mathcal{F}$ and $\tau > 0$.

(2). There are $x_0, y_0, z_0 \in \mathbb{R}_+$ such that $x_0 \leq C(y_0), y_0 \geq C(z_0)$.

Then, three person game $\mathcal{G}$ has a non- cooperative equilibrium.

Now we shall give an example to show that Corollary 4.2 is effective.

Example 4.3. *Consider $S = \mathbb{R}_+$ endowed with the metric $d(x, y) = |x - y|$ for all $x, y \in S$. Let $\mathcal{G}$ be a three person game with triloss operator*

$$L_1(s_1, s_2, s_3) = s_1^2(1 + s_2 + s_3)e^{-\tau} - 3s_1$$

$$L_2(s_1, s_2, s_3) = s_2^2(1 + s_1 + s_3)e^{-\tau} - 3s_2$$

$$L_3(s_1, s_2, s_3) = s_3^2(1 + s_1 + s_2)e^{-\tau} - 3s_3$$

where $(s_1, s_2, s_3) \in \mathbb{R}_+^3$ and a given $\tau > 0$. It is easy to compute that the optimal decision rules C_1, C_2, C_3 such that of $\mathcal{G}$

$$C_1(s_3) = \frac{e^{-\tau}}{1 + s_3},$$
$$C_2(s_2) = \frac{e^{-\tau}}{1 + s_2},$$
$$C_3(s_1) = \frac{e^{-\tau}}{1 + s_1},$$

where $s_1, s_2, s_3 \in \mathbb{R}_+$. We have $C_1(s) = C_2(s) = C_3(s)$ for all $s \in \mathbb{R}_+$, and C is continuous map. We need show that C satisfies all conditions of Corollary 4.2. We have

$$d(C(x), C(y)) = e^{-\tau}\Big|\frac{1}{1 + x} - \frac{1}{1 + y}\Big| \leq e^{-\tau}|x - y| = e^{-\tau}d(x, y)$$

for all $x, y \in \mathbb{R}_+$*. By passing to logarithms, we arrive at*

$$\tau + \ln d(C(x), C(y)) \leq \ln d(x, y)$$

for all $x \neq y$*. Since* $F(x) = \ln x \in \mathcal{F}$ *we can deduce that* C *satisfies 1) in Corollary 4.2. Choosing* $x_0 = 0$*, we have that*

$$C(x_0) = e^{-\tau}.$$

Let $y_0 = 1$*, we have* $y_0 \geq C(x_0)$*. On the other hand* $x_0 = 0 \leq C(y_0) = \dfrac{e^{-\tau}}{3}$*. Therefore,* C *satisfies all conditions of Corollary 4.2. Applying this corollary, we get that three person game* $\mathcal{G}$ *has a non- cooperative equilibrium.*

5. Application to Nonlinear Integral Equations

In this section, we study the existence of unique solution of nonlinear integral equations, as an application of the fixed point theorem proved in Section 3.

Let us consider the following integral equation

$$x(t) = h(t) + \int_0^t [K_1(t,s) + K_2(t,s) + K_3(t,s)]\big(f(s, x(s)) + g(s, x(s) + j(s, x(s))\big)ds, \tag{36}$$

where the unknown functions $x(t)$ takes the real values.

Let $X = C([0, K])$ be the space of all real continuous functions defined on $[0, K]$. It well known that $C([0, K])$ endowed with the metric

$$d(x, y) = \|x - y\| = \max_{t \in [0,K]} |x(t) - y(t)|$$

is a complete metric space. By a solution of the (36), we mean a continuous function $x \in X$ that satisfies the equation (36) on $[0, K]$. By certain conditions on K_1, K_2, K_3, f, g, j and using the results of previous section, we will prove that (36) has a unique solution. For this, note that X can be equipped with the partial order $\preceq$ given by

$$x, y \in X, \quad x \preceq y \Longleftrightarrow \big(x(t) \leq y(t)\ \forall t \in [0, K] \text{ and } \|x\|, \|y\| \leq 1\big) \text{ or } x(t) = y(t)\ \forall\, t \in [0, K]. \tag{37}$$

We assume that the functions K_1, K_2, K_3, f, g, j fulfill the following conditions.

Assumption 5.1.

(A). $f, g, j \in C([0, K] \times \mathbb{R})$*,* $h \in X$ *and* $K_1, K_2, K_3 \in C([0, K] \times [0, K])$ *such that* $K_1(t, s) \geq 0$*,* $K_2(t, s) \leq 0$ *and* $K_3(t, s) \geq 0$ *for all* $t, s \geq 0$*;*

(B). $f(t, .), j(t, .) : \mathbb{R} \to \mathbb{R}$ *is increasing for all* $t \in [0, K]$*;* $g(t, .) : \mathbb{R} \to \mathbb{R}$ *is decreasing for all* $t \in [0, K]$*;*

(C). There exist $\tau \in [1, \infty)$ *such that*

$$0 \leq f(t, x) - f(t, y) \leq \tau e^{-\tau} \frac{x - y}{3},\ \forall x \geq y,$$

$$-\tau e^{-\tau} \frac{x - y}{3} \leq g(t, x) - g(t, y) \leq 0,\ \forall x \geq y$$

$$\text{and}\ \ 0 \leq j(t, x) - j(t, y) \leq \tau e^{-\tau} \frac{x - y}{3},\ \forall x \geq y$$

(D). $\{\max_{t,s\in[0,K]}|K_1(t,s)-K_2(t,s)|, \max_{t,s\in[0,K]}|K_2(t,s)-K_3(t,s)|, \max_{t,s\in[0,K]}|K_3(t,s)-K_1(t,s)|\}\leq 1.$

Define $T: X\times X\times X\to X$ by

$$\begin{aligned}T(x,y,z)(t)=h(t)&+\int_0^t K_1(t,s)\big(f(s,x(s))+g(s,y(s)+j(s,z(s))\big)ds\\&+\int_0^t K_2(t,s)\big(f(s,y(s))+g(s,x(s))+j(s,y(s))\big)ds\\&+\int_0^t K_3(t,s)\big(f(s,z(s))+g(s,y(s))+j(s,x(s))\big)ds\end{aligned}$$

for all $t\in[0,K]$.

Definition 5.2. *An element $(\alpha,\beta,\gamma)\in C([0,K]\times C[0,K]\times C[0,K])$ is a tripled normal lower and a normal upper solution of the integral equation (36) if $\alpha\preceq\beta$ and $\alpha\preceq T(\alpha,\beta,\gamma),\beta\succeq T(\beta,\alpha,\beta)$ and $\gamma\preceq T(\gamma,\beta,\alpha)$.*

Theorem 5.3. *Suppose that Assumption 5.1 is fulfilled. Then the existence of a Tripled normal lower and normal upper solution for (36) provides the existence of a unique solution of (36) in $C([0,K])$.*

Proof. Suppose $\{u_n\}$ is a monotone non-decreasing sequence in X that converges to $u\in X$. Then for every $t\in[0,K]$, the sequence of real numbers $u_1(t)\leq u_2(t)\leq ...\leq u_n(t)\leq ...$ converges to $u(t)$. Moreover, since the normed map is continuous, we can deduce that $\|u\|\leq 1$ if provided $\|u_n\|\leq 1$ for all n. Therefore, for every $t\in[0,K]$, $n\in\mathbb{N}$, $u_n(t)\leq u(t)$. Hence $u_n\leq u$, for all $n\in\mathbb{N}$.

Similarly, we can verify that limit $v(t)$ of a monotone non-increasing sequence $v_n(t)$ in X is a lower bound for all elements in the sequence. That is, $v\leq v_n$ for all n. Hence, the condition (b) in Corollary 3.3 holds. Also we can verify that limit $w(t)$ of a monotone non-decreasing sequence $w_n(t)$ in X is a upper bound for all elements in the sequence. That is, $w\geq w_n$ for all n. Hence, the condition (b) in Corollary 3.3 holds. For $x\in X$, we defined $\|x\|_\tau=\max_{t\in[0,K]}|x(t)|e^{-\tau t}$, where $\tau\geq 1$ is chosen arbitrary. It is easy to check that $\|.\|_\tau$ is a norm equivalent to the maximum norm in X and X endowed with the metric d_τ defined by

$$d_\tau(x,y)=\|x-y\|_\tau=\max_{t\in[0,K]}\{|x(t)-y(t)|e^{-\tau t}\}$$

for all $x,y\in X$ is a complete metric space. Now, consider X endowed with partial metric given by

$$p_\tau(x,y)=\begin{cases}d_\tau(x,y) & \text{if } \|x\|_\tau\leq 1,\|y\|_\tau\leq 1\\ d_\tau(x,y)+\tau & \text{otherwise.}\end{cases}\tag{38}$$

It is easy to see that (X,p_τ) is a 0-complete partial metric space but is not complete (See [22]). We recall that $T: X\times X\times X\to X$ by

$$\begin{aligned}T(x,y,z)(t)=\ h(t)&+\int_0^t K_1(t,s)\big(f(s,x(s))+g(s,y(s)+j(s,z(s))\big)ds\\&+\int_0^t K_2(t,s)\big(f(s,y(s))+g(s,x(s))+j(s,y(s))\big)ds\\&+\int_0^t K_3(t,s)\big(f(s,z(s))+g(s,y(s))+j(s,x(s))\big)ds\end{aligned}$$

for all $t\in[0,K]$. Next, we show that T has the mixed monotone property. Indeed, for $x_1,x_2\in C([0,K])$ and $x_1\leq x_2$ that is $x_1(t)\leq x_2(t)$ for every $t\in[0,K]$, we have

$$T(x_1,y,z)(t)-T(x_2,y,z)(t)=\int_0^t K_1(t,s)\big[f(s,x_1(s))+g(s,y(s))+j(s,z(s))\big]ds$$

$$
\begin{aligned}
&+\int_0^t K_2(t,s)\big[f(s,y(s))+g(s,x_1(s))+j(s,y(s))\big]ds\\
&+\int_0^t K_3(t,s)\big[f(s,z(s))+g(s,y(s))+j(s,x_1(s))\big]ds+h(t)\\
&-\int_0^t K_1(t,s)\big[f(s,x_2(s))+g(s,y(s))+j(s,z(s))\big]ds\\
&-\int_0^t K_2(t,s)\big[f(s,y(s))+g(s,x_2(s))+j(s,y(s))\big]ds\\
&-\int_0^t K_3(t,s)\big[f(s,z(s))+g(s,y(s))+j(s,x_2(s))\big]ds-h(t)\\
&=\int_0^t K_1(t,s)\big[f(s,x_1(s))-f(s,x_2(s))\big]ds\\
&+\int_0^t K_2(t,s)\big[g(s,x_1(s))-g(s,x_2(s))\big]ds\\
&+\int_0^t K_3(t,s)\big[j(s,x_1(s))-j(s,x_2(s))\big]ds\\
&\le 0
\end{aligned}
$$

for every $t \in [0, K]$, by Assumption 5.1. This yields $T(x_1, y)(t) \le T(x_2, y)(t)$ for every $t \in [0, K]$, that is $T(x_1, y, z) \le T(x_2, y, z)$. By the same computation, we arrive at $T(x, y_1, z) \le T(x, y_2, z)$ if $y_1 \ge y_2$ and $T(x, y, z_1) \le T(x, y, z_2)$ if $z_1 \le z_2$. Hence, T has the mixed monotone property.

Now, for $x \ge u$, $y \le v$ and $z \le w$, we have

$$
\begin{aligned}
|(T(x,y,z)(t)-T(u,v,w)(t)| &= \Big|\Big[\int_0^t K_1(t,s)\big(f(s,x(s))+g(s,y(s))+j(s,z(s))\big)ds\\
&+\int_0^t K_2(t,s)\big(f(s,y(s))+g(s,x(s))+j(s,x(s))\big)ds\\
&+\int_0^t K_3(t,s)\big(f(s,z(s))+g(s,y(s))+j(s,x(s))\big)ds+h(t)\Big]\\
&-\Big[\int_0^t K_1(t,s)\big(f(s,u(s))+g(s,v(s))+j(s,w(s))\big)ds\\
&+\int_0^t K_2(t,s)\big(f(s,v(s))+g(s,u(s))+j(s,v(s))\big)ds\\
&+\int_0^t K_3(t,s)\big(f(s,w(s))+g(s,v(s))+j(s,u(s))\big)ds+h(t)\Big]\Big|\\
&=\Big|\int_0^t K_1(t,s)\big[\big(f(s,x(s))-f(s,u(s))\big)\\
&+\big(g(s,y(s))-g(s,v(s))\big)+\big(j(s,z(s))-j(s,w(s))\big)\big]ds\\
&+\int_0^t K_2(t,s)\big[\big(f(s,y(s))-f(s,v(s))\big)\\
&+\big(g(s,x(s))-g(s,u(s))\big)+\big(j(s,y(s))-j(s,v(s))\big)\big]ds\\
&+\int_0^t K_3(t,s)\big[\big(f(s,z(s))-f(s,w(s))\big)\\
&+\big(g(s,y(s))-g(s,v(s))\big)+\big(j(s,x(s))-j(s,u(s))\big)\big]ds|
\end{aligned}
$$

$$
\begin{aligned}
&=\Big|\int_0^t K_1(t,s)\big[\big(f(s,x(s))-f(s,u(s))\big)-\big(g(s,v(s))-g(s,y(s))\big)-\big(j(s,z(s))-j(s,w(s))\big)\big]ds\\
&-\int_0^t K_2(t,s)\big[\big(f(s,v(s))-f(s,y(s))\big)-\big(g(s,x(s))-g(s,u(s))\big)-\big(j(s,v(s))-j(s,y(s))\big)\big]ds\Big|\\
&\le\Big|\int_0^t K_1(t,s)\tau e^{-\tau}\big[\frac{x(s)-u(s)}{3}+\frac{v(s)-y(s)}{3}+\frac{z(s)-w(s)}{3}\big]ds\\
&-\int_0^t K_2(t,s)\tau e^{-\tau}\big[\frac{v(s)-y(s)}{3}+\frac{x(s)-u(s)}{3}+\frac{v(s)-y(s)}{3}\big]ds
\end{aligned}
$$

$$
\begin{aligned}
&- \int_0^t K_3(t,s)\tau e^{-\tau}\big[\frac{z(s)-w(s)}{3}+\frac{v(s)-y(s)}{3}+\frac{x(s)-u(s)}{3}\big]ds\Big| \\
&\le \tau e^{-\tau}\int_0^t \big|[K_1(t,s)-K_2(t,s)-K_3(t,s)]\Big[\frac{x(s)-u(s)}{3}+\frac{v(s)-y(s)}{3}+\frac{z(s)-w(s)}{3}\Big]\Big|ds \\
&= \tau e^{-\tau}\int_0^t \big|K_1(t,s)-K_2(t,s)-K_3(t,s)\big|e^{\tau s}\Big[\frac{|x(s)-u(s)|e^{-\tau s}}{3}+\frac{|v(s)-y(s)|e^{-\tau s}}{3}+\frac{|z(s)-w(s)|e^{-\tau s}}{3}\Big]ds \\
&\le \tau e^{-\tau}\int_0^t \max_{t,s\in[0,K]}\big|K_1(t,s)-K_2(t,s)-K_3(t,s)\big|e^{\tau s}\Big[\frac{\|x-u\|_\tau}{3}+\frac{\|y-v\|_\tau}{3}\frac{\|z-w\|_\tau}{3}\Big]ds \\
&\le \tau e^{-\tau}\frac{e^{\tau t}}{\tau}\Big[\frac{\|x-u\|_\tau}{3}+\frac{\|y-v\|_\tau}{3}+\frac{\|z-w\|_\tau}{3}\Big].
\end{aligned}
$$

It follows that

$$
|T(x,y,z)(t)-T(u,v,w)(t)|e^{-\tau t}\le e^{-\tau}\Big[\frac{\|x-u\|_\tau}{3}+\frac{\|y-v\|_\tau}{3}+\frac{\|z-w\|_\tau}{3}\Big].
$$

Hence, for all $x,y,z,u,v,w\in X$ such that $x\ge u$, $y\le v$ and $z\ge w$, since $\|x\|_\tau,\|y\|_\tau,\|z\|_\tau,\|u\|_\tau,\|v\|_\tau,\|w\|_\tau\le 1$, we have

$$
p_\tau\big(T(x,y,z),T(u,v,w)\big)\le e^{-\tau}\frac{1}{3}\Big[p_\tau(x,u)+p_\tau(y,v)+p_\tau(z,w)\Big]
$$

By passing to logarithms, we arrive at

$$
\tau+\ln p_\tau\big(T(x,y,z),T(u,v,w)\big)\le \ln\Big(\frac{p_\tau(x,u)+p_\tau(y,v)+p_\tau(z,w)}{3}\Big).
$$

Since $F(x)=\ln x\in\mathcal{F}$, we conclude that T satisfies the condition (26). Now, let us (α,β,γ) be a tripled normal lower and normal upper solution of the integral equation of (36). Then, we have $\alpha\preceq\beta$, $\beta\succeq\gamma$

$$
\alpha\preceq T(\alpha,\beta,\gamma),\beta\succeq T(\beta,\alpha,\beta)\text{ and }\gamma\preceq T(\gamma,\beta,\alpha).
$$

Finally, applying Corollary 3.4, we can conclude that T has a fixed point x. Hence $T(x,x,x)=x$ and x is an unique solution of the equation (36). □

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