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RESEARCH ARTICLE

NEW COMMON FIXED POINT RESULTS FOR DIGITAL TYPE EXPANSIVE MAPPINGS IN DIGITAL METRIC SPACE

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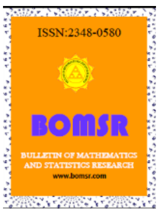


ABSTRACT

Information Technology plays a vital role in every step of human activity in the 21st century. In this era, we can't imagine our life without technology because the virtual communication takes place over traditional communication by using various ways like video conferencing, telephoning, Emailing, Image processing, etc. On the other hand fixed point theory is a very traditional topic of pure part of Mathematics which plays a vital role in many disciplinary of Mathematics such as Topology, Physics, Chemistry, Biology, Computer Science, etc. In the second half years of the 20th century, researchers tried to provide a digital version of this theory. A digital image is the form of pixels and we can consider the distance between pixels by a distance function. As the distance between pixels is very least then we find a good image quality while the distance is maximum then we obtain a poor quality image. In this paper, we established existence of some new fixed point results by taking two self-mappings for expansive mappings in digital metric space.

Keywords: Fixed Point, Digital image, Digital type expansive mappings

Introduction

The fixed point theory (FPT) is now a promising area in mathematics especially in nonlinear functional analysis because it has wide applicability in various fields of pure and applied mathematics as well as in other fields like Physical science, Life science Economics, etc. The field of the FPT is expanding its domain, thereby leading to the emergence of a plethora of techniques and ideas. The fixed point theory is one of the very effective and fruitful tools in Mathematics which has huge applications inside as well as outside mathematics. It is an influential field of specialization and has grown into a full branch of Mathematics within more than a hundred years out of multitude problems occurring in diversified fields. It is very difficult to imagine its applications in many fields.

Nowadays the human world has become completely digital. In every step of life humans use technology in every sector such as Medical, Education, Defence, etc. From these Image processing is a promising area in which FPT is more applicable.

Fixed Point theory for digital image firstly studied by A. Rosenfield [10]. In 2017, Jyoti [6] proved a fixed point theorem for expansive mapping on complete digital metric space. In recent years, many authors discussed on fixed point result in view of digital images and satisfy certain digital type contractive and expansive conditions with applications [1,3,5,6,7,8,9,11,12]. In this paper we establish new fixed point results for two self mappings satisfy digital type expansive condition in digital metric space. I hope this result extends many well known results. But, before proving the main result we need to explain some prerequisites.

2. PREREQUISITES:

Definition 2.1.[1]: Suppose Z^n the set of all integers defined by $Z^n = \{(x_1, x_2, x_3, \cdots x_n) : x_i \in Z, 1 \le i \le n\}$ and called as the set of all lattice point in the n-dimensional Euclidean space. Suppose $X \subseteq Z^n \ \forall n \in Z$ and α represent and adjacency relation for the members of X then (X, α) is called **digital image.**

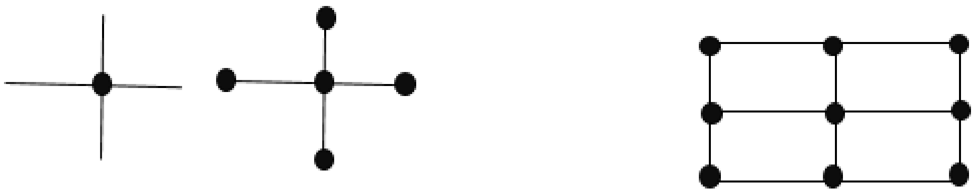
Definition2.2.[1]: Assume $p, q \in Z^n$ Where $p = (p_1, p_2, p_3 \cdots p_n)$ and $q = (q_1, q_2, q_3 \cdots q_n)$ Let $n \in Z$ such that $1 \le \alpha \le n$ then we say p and q are $\alpha - adjacent$ in Z^n if there exist at most α indices i such that $|p_i - q_i| \le 1$ and for all other indices j such that $|p_j - q_j| \ne 1$ we obtain $p_j - q_j$. From this definition following statement can be obtained

For given $p \in Z^n$ the number of points $q \in Z^n$ which are $\alpha - adjacent$ to p is denoted by $K(\alpha, n)$ Where $K(\alpha, n)$ is independent to p.

For our convience we write $K = K(\alpha, n)$, then we have following cases

1) If $n = 1$ then assume $p \in Z$ then α will have only one value $\alpha = 1$. Then in this case $K = K(1,1) = 2$. So in this case $p - 1$ and $p + 1$ only two points are 1- adjacent to $p \in Z$ also q is 1-adjacent to p iff $|p - q| = 1$

2) Let $n = 2$ then assume $p \in Z^2{}_{++}$ so α will have $\alpha = 1,2$ if $\alpha = 2$ then 2 adjacent to $p = (p_1, p_2)$. In this case $(p_1 \pm 1, p_2)$ $(p_1, p_2 \pm 1)$ $(p_1 \pm 1, p_2 \pm 1)$ $(p_1 + 1, p_2 \pm 1)$ are only points of 2- adjacent to p such that $K = K(2,2) = 8$



1-adjacency **4-adjacency** **8-adjacency**

Similarly $n = 3$ then. $K = K(3,3) = 26$. In general for study n-D digital image. If $1 \le \alpha \le n$ then $K = K(\alpha, n)$ is given by

$$K(\alpha, n) = \sum_{i=n-\alpha}^{n-1} 2^{n-i} C_i^n \qquad C_i^n = \frac{n!}{(n-i)!i!}$$

Definition 2.3. [1]: Let $(X,k) \subseteq Z^n$ be a digital image. Define a function $d : X \times X \to [0, \infty)$ by,

$$d(p,q) = \left[\sum_{i=1}^{n} (p_i - q_i)^2\right]^{\frac{1}{2}}$$

Then we have the following properties satisfied by *d* for all x, y, z ∈ X.

DMS_1 $\quad d(x,y) \ge 0$

DMS_2 $\quad d(x,y) = 0 \Leftrightarrow x = y$

DMS_3 $\quad d(x,y) = d(y,x)$

DMS_4 $\quad d(x,y) \le d(x,z) + d(z,y)$

The digital image *(X, k)* together with the function *d* is called a digital metric space with *k*-adjacency. It is denoted by (*X, d, k).*

i.e.***"A metric space equipped with digital image is called digital metric space."***

Definition 2.4. [1]: Let (*X*, *d*,k) be a digital metric space then sequence $\{x_n\}$ of the points of (*X, d*,k) is called

2.4.1.Cauchy sequences in*X* if $\forall \varepsilon > 0, \exists\, n_0 \in Z_+$ such that $d(x_n, x_m) < \varepsilon, \forall\, n, m > n_0$.

2.4.2.converges to a limit L in *X* if for given $\varepsilon > 0, \exists\, n_0 \in Z_+$ such that $d(x_n, L) < \varepsilon, \forall\, n, > n_0$. 2.4.3. **complete** if every Cauchy sequence $\{x_n\}$ of point of(*X, d*,k) converges to a point *L* of (*X, d*,k).

Definition 2.5[1]: Let (X, d, k) be a digital metric space and $T: X \to X$ be a mapping then a point $x \epsilon X$ is called a digital fixed point of T if x is mapped into itself, i.e.$T(x) = x$.

Theorem [6]: Let $T : (X, d, k) \to (X, d, k)$ be a mapping on a complete digital metric space *X.* Let *T* be onto and satisfy $d(Tx, Ty) \ge \lambda d(x, y)\ \forall x \in X$ and $\lambda > 1$. Then *T* has a fixed point in *X.* In this condition *T* is known as expansive mapping

3. MAIN RESULTS

In this section we proof five theorems with different type digital expansive conditions

Theorem 3.3.1: Let (X,k) be a digital image where $X \subseteq Z^n$ and *k* is an adjacency relation in *X.* Let (X,d,k) be a complete digital metric space and suppose $T_1, T_2 : X \to X$ be any two continuous and onto mapping satisfying the condition

$$d(T_1x, T_2y) \geq \alpha d(x,y) + \beta[d(x,T_1x) + d(y,T_2y)]$$

for all *x, y* ∈ X, where $\alpha > 0, \beta \in \left[\frac{1}{2}, 1\right]$ are constant, with $\alpha + \beta > 1$. Then $T_1$and T_2have a common fixed point in *X.*

Proof: Let $x_0 \in X$ is arbitrary point. Since $T_1$and T_2be onto so there exist $x_0 \in X$ and $x_1 \in X$ such that $T_1(x_1) = x_0, T_2(x_2) = x_1$.

In this way, we define the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ by

$x_{2n} = T_1x_{2n+1}$for $n = 0,1,2,3$ …….and

$x_{2n+1} = T_2x_{2n+2}$ for $n = 0,1,2,3 \dots \dots$

Now put $x = x_{2n+1}$and $y = x_{2n+2}$, we have

$d(x_{2n}, x_{2n+1}) = d(T_1x_{2n+1}, T_2x_{2n+2})$

$\Rightarrow d\, x_{2n}, x_{2n+1} \geq \alpha d(x_{2n+1}, x_{2n+2}) + \beta[d(x_{2n+1}, T_1x_{2n+1}) + d(x_{2n+2}, T_2x_{2n+2})]$

$\geq \alpha d(x_{2n+1}, x_{2n+2}) + \beta[d(x_{2n+1}, x_{2n}) + d(x_{2n+2}, x_{2n+1})]$

$= (\alpha+\beta)\, d(x_{2n+1}, x_{2n+2}) + \beta d(x_{2n+1}, x_{2n})$

$\Rightarrow (1-\beta)d(x_{2n}, x_{2n+1}) \geq (\alpha+\beta)\, d(x_{2n+1}, x_{2n+2})$

$\Rightarrow d(x_{2n}, x_{2n+1}) \geq \frac{\alpha+\beta}{1-\beta} d(x_{2n+1}, x_{2n+2})$

$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq \frac{1-\beta}{\alpha+\beta} d(x_{2n}, x_{2n+1})$

$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq hd(x_{2n}, x_{2n+1}), \quad where\ h = \frac{1-\beta}{\alpha+\beta},\ 0 \leq h \leq 1.$

In general

$d(x_{2n}, x_{2n+1}) \leq hd(x_{2n-1}, x_{2n}) \leq \dots\dots\dots \leq h^{2n}d(x_0, x_1)$

So, for $n < m$we have

$d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n+2}) + \dots\dots\dots d(x_{2m-1}, x_{2m})$

$\leq (h^{2n} + h^{2n+1} + \cdots\cdots\cdots + h^{2m-1})d(x_0, x_1)$

$= \frac{h^{2n}}{1-h} d(x_0, x_1)$

Let $c \geq 0$ be given, choose a natural number *N* such that $\frac{h^{2n}}{1-h} d(x_0, x_1) \leq c$, for all $n \geq N_1$. Thus $d(x_{2n}, x_{2m}) \leq c$, for $m > n$. Therefore$\{x_{2n}\}$ is a Cauchy sequence in (X,d,k).

Since (X, d, k) is a complete digital metric space, there exist $z \in X$such that$x_{2n} \to z$ as $n \to \infty$. Also if T_1 is continuous, then

$$d(T_1 z, z) \leq d(T_1 x_{2n+1}, T_1 z) + d((T_1 x_{2n+1}, z) \to 0 as n \to \infty.$$

$Since \quad x_{2n} \to z \; and \quad T_1 x_{2n+1} \to T_1 z \to z \; as \quad n \to \infty$. Therefore $d(T_1 z, z) =$ 0. This implies that, $T_1 z$ = z. Hence z is a fixed point ofT_1

Similarly, it can be established that $T_2 z$ = z .Therefore, $T_1 z$= z=, $T_2 z$. Thus $z$is the common fixed point of pair of maps T_1 andT_2.This completes the proof.

Theorem 3.2: Let (X, k) be a digital image where $X \subseteq Z^n$ and *k* is an adjacency relation in *X.* Let (X, d, k) be a complete digital metric space and suppose $T_1, T_2 : X \to X$ be any two continuous and onto mapping satisfying the condition

$$d(T_1 x, T_2 y) \geq \alpha d(x, y) + \beta[d(x, T_2 y) + d(y, T_1 x)] \ldots\ldots\ldots\ldots 3.1.2$$

for all *x, y* ∈ X, where$\alpha > 0$, $½ \leq \beta \leq 1$are constant, with $\alpha + \beta > 1$. Then $T_1$and T_2have a common fixed point in *X.*

Proof: Let x_o be an arbitrary point in X. Since $T_1$and T_2be onto (surjective), there exist $x_o \in X$ and $x_1 \in X$ such that

$$T_1(x_1) = x_0, T_2(x_2) = x_1$$

In this way, we define the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ by

$x_{2n} = T_1 x_{2n+1}$for $n = 0,1,2,3$ …….

and$x_{2n+1} = T_2 x_{2n+2}$ for $n = 0,1,2,3 \ldots$ ….

Note that, If $x_{2n} = x_{2n+1}$ some n ≥ 1, then it is fixed point of $T_1$and T_2.

Now putting $x = x_{2n+1}$and $y = x_{2n+2}$, we have

$$d(x_{2n}, x_{2n+1}) = d(T_1 x_{2n+1}, T_2 x_{2n+2})$$

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &\geq \alpha d(x_{2n+1}, x_{2n+2}) + \beta[d(x_{2n+1}, T_2 x_{2n+2}) + d(x_{2n+2}, T_1 x_{2n+1})] \\ &\geq \alpha d(x_{2n+1}, x_{2n+2}) + \beta[d(x_{2n+1}, x_{2n+1}) + d(x_{2n+2}, x_{2n})] \\ &\geq \alpha d(x_{2n+1}, x_{2n+2}) + \beta[d(x_{2n+2}, x_{2n+1}) + d(x_{2n+1}, x_{2n})] \\ &= (\alpha + \beta)\, d(x_{2n+1}, x_{2n+2}) + \beta d(x_{2n+1}, x_{2n}) \end{aligned}$$

$\Rightarrow (1 - \beta) d(x_{2n}, x_{2n+1}) \geq (\alpha + \beta)\, d(x_{2n+1}, x_{2n+2})$

$\Rightarrow d(x_{2n}, x_{2n+1}) \geq \frac{\alpha+\beta}{1-\beta} d(x_{2n+1}, x_{2n+2})$

$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq \frac{1-\beta}{\alpha+\beta} d(x_{2n}, x_{2n+1})$

$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq h d(x_{2n}, x_{2n+1}), \quad where \; h = \frac{1-\beta}{\alpha+\beta}, \; 0 \leq h \leq 1.$

In general

$d(x_{2n}, x_{2n+1}) \leq h d(x_{2n-1}, x_{2n}) \leq \ldots\ldots\ldots h^{2n} d(x_0, x_1)$

So, for$n < m$,we have

$d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n+2}) + \dots\dots\dots d(x_{2m-1}, x_{2m})$

$\leq (h^{2n} + h^{2n+1} + \cdots\cdots\cdots\cdots + h^{2m-1}) d(x_0, x_1)$

$\leq \frac{h^{2n}}{1-h} d(x_0, x_1)$

Let $0 \leq c$ be given, choose a natural number N_1 such that $\leq \frac{h^{2n}}{1-h} d(x_0, x_1) \leq c$, for all $n \geq N_1$. Thus $d(x_{2n}, x_{2m}) \leq c$, for $n < m$.Therefore$\{x_{2n}\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete digital metric space, there exist $z \in X$such that $x_{2n} \to z$ as $n \to \infty$. If T_1 is continuous, then

$$d(T_1 z, z) \leq d(T_1 x_{2n+1}, T_1 z) + d((T_1 x_{2n+1}, z) \to 0 as n \to \infty.$$

$Since \quad x_{2n} \to z \, and \quad T_1 x_{2n+1} \to T_1 z \, as \quad n \to \infty$. Therefore $d(T_1 z, z) = 0$. This implies that, $T_1 z$ = z. Hence z is a fixed point ofT_1

Similarly, it can be established that $T_2 z$ = z .Therefore, $T_1 z$= z =, $T_2 z$. Thus $z$is the common fixed point of pair of maps T_1 andT_2 .This completes the proof.

Theorem 3.3: Let (X, k) be a digital image where $X \subseteq Z^n$ and k is an adjacency relation in X. Let (X, d, k) be a complete digital metric space and suppose $T_1, T_2 : X \to X$ be any two continuous and onto mapping satisfying the condition

$d(T_1 x, T_2 y) \geq \alpha d(x, y) + \beta d(x, T_1 x) + \gamma d(y, T_2 y) + \eta[d(x, T_1 x) + d(y, T_2 y)]$……3.3.1

for all x, $y \in X$, where$\alpha \geq -1, \beta > 0, \gamma \leq ½ \, and \, ½ < \eta \leq 1$ are constant, with $\alpha + \beta + \gamma + \eta > 1$. Then $T_1$and T_2have a common fixed point in X.

Proof: Let x_o be an arbitrary point in X. Since $T_1$and T_2be onto (surjective), there exist $x_o \in X$ and $x_1 \in X$ such that

$$T_1(x_1) = x_0, \;\; T_2(x_2) = x_1$$

In this way, we define the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ by

$$x_{2n} = T_1 x_{2n+1} \text{for} \quad n = 0, 1, 2, 3 \dots\dots$$

and $$x_{2n+1} = T_2 x_{2n+2} \text{ for} \quad n = 0, 1, 2, 3 \dots \dots$$

Now putting $x = x_{2n+1}$and $y = x_{2n+2}$.

Then we have,

$d(x_{2n}, x_{2n+1}) = d(T_1 x_{2n+1}, T_2 x_{2n+2})$

$\Rightarrow d(x_{2n}, x_{2n+1}) \geq \alpha d(x_{2n+1}, x_{2n+2}) + \beta d(x_{2n+1}, T_1 x_{2n+1}) + \gamma d(x_{2n+2}, T_2 x_{2n+2})$

$+\eta[(x_{2n+1}, T_1 x_{2n+1}) + (x_{2n+2}, T_2 x_{2n+2})]$

$\geq \alpha d(x_{2n+1}, x_{2n+2}) + \beta d(x_{2n+1}, x_{2n}) + \gamma \, d(x_{2n+2}, x_{2n+1})$

$+\eta[(x_{2n+1}, x_{2n}) + (x_{2n+2}, x_{2n+1})]$

$= (\alpha + \gamma + \eta) d(x_{2n+1}, x_{2n+2}) + (\beta + \eta) \, d(x_{2n+1}, x_{2n})$

$\Rightarrow [1 - (\beta + \eta)] d(x_{2n}, x_{2n+1}) \geq ((\alpha + \gamma + \eta) \, d(x_{2n+1}, x_{2n+2})$

$\Rightarrow d(x_{2n}, x_{2n+1}) \geq \frac{\alpha+\gamma+\eta}{1-(\beta+\eta)} d(x_{2n+1}, x_{2n+2})$

$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq \frac{1-(\beta+\eta)}{\alpha+\gamma+\eta} d(x_{2n}, x_{2n+1})$

$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq hd(x_{2n}, x_{2n+1}), where\ h = \frac{1-(\beta+\eta)}{\alpha+\gamma+\eta},\ 0 \leq h \leq 1.$

In general

$d(x_{2n}, x_{2n+1}) \leq hd(x_{2n-1}, x_{2n}) \leq \ldots, \ldots, \ldots. h^{2n} d(x_{0,} x_1)$

So, for $n < m$,we have

$$d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n+2}) + \ldots, \ldots, \ldots d(x_{2m-1}, x_{2m})$$
$$\leq (h^{2n} + h^{2n+1} + \cdots\cdots\cdots + h^{2m-1}) d(x_0, x_1)$$
$$\leq \frac{h^{2n}}{1-h} d(x_0, x_1)$$

Let $0 \leq c$ be given, choose a natural number N_1 such that $\frac{h^{2n}}{1-h} d(x_0, x_1) \leq c$, for all $n \geq N_1$. Thus $d(x_{2n}, x_{2m}) \leq c$, for $n < m$.Therefore$\{x_{2n}\}$ is a Cauchy sequence in (X, d, k). Since (X, d, k) is a complete digital metric space, there exist $z \in X$such that$x_{2n} \to z$ as $n \to \infty$.

If T_1 is continuous, then

$$d(T_1 z, z) \leq d(T_1 x_{2n+1}, T_1 z) + d((T_1 x_{2n+1}, z) \to 0 as n \to \infty.$$

$Since\ x_{2n} \to z\ and\ T_1 x_{2n+1} \to T_1 z\ as\ n \to \infty$. Therefore $d(T_1 z, z)$ = 0. This implies that, $T_1 z$ = z. Hence z is a fixed point ofT_1

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Theorem 3.4: Let (X, k) be a digital image where $X \subseteq Z^n$ and k is an adjacency relation in X. Let (X, d, k) be a complete digital metric space and suppose $T_1, T_2 : X \to X$ be any two continuous and onto mapping satisfying the condition

$d(T_1 x, T_2 y) \geq \alpha[d(x, y) + d(x, T_1 x) + d(y, T_2 y)] + [\beta d(x, T_2 y) + d(y, T_1 x)]$ ...3.1.4

for all *x, y* ∈ X, where$\alpha \geq 0, \beta < 1$ are constant, with $\alpha + \beta > 1$. Then $T_1$and T_2have a common fixed point in *X.*

proof : Let x_o be an arbitrary point in X. Since $T_1$and T_2be onto (surjective), there exist $x_o \in X$ and $x_1 \in X$ such that

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In this way, we define the sequences $\{x_{2n}\}$ and$\{x_{2n+1}\}$ by

$x_{2n} = T_1 x_{2n+1}$for $n = 0,1,2,3$ ....

and $x_{2n+1} = T_2 x_{2n+2}$ for $n = 0,1,2,3 \ldots$ ....

Note that, If $x_{2n} = x_{2n+1}$ some n ≥ 1, then it is fixed point of $T_1$and T_2.

Now putting $x = x_{2n+1}$and $y = x_{2n+2}$, we have

$$d(x_{2n}, x_{2n+1}) = d(T_1 x_{2n+1}, T_2 x_{2n+2})$$
$$\geq \alpha[d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, T_1 x_{2n+1}) + d(x_{2n+2}, T_2 x_{2n+2})]$$

$$+\beta[(x_{2n+1}, T_2x_{2n+2}) + \quad (x_{2n+2}, T_1x_{2n+1})]$$

$$\geq \alpha[d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, x_{2n}) + d(x_{2n+2}, x_{2n+1})]$$

$$+\beta[(x_{2n+1}, x_{2n+1}) + (x_{2n+2}, x_{2n})]$$

$$\geq \alpha[2d(x_{2n+1}, x_{2n+2}) + d(x_{2n+1}, x_{2n})])$$

$$+\beta[d(x_{2n+1}, x_{2n+2} + d(x_{2n+2}, x_{2n+1}) + d(x_{2n+1}, x_{2n})]$$

$$= (2\alpha + 2\beta)d(x_{2n+1}, x_{2n+2}) + (\alpha + \beta)d(x_{2n}, x_{2n+1})$$

$\Rightarrow[1 - (\alpha + \beta)]d(x_{2n}, x_{2n+1}) \geq (2\alpha + 2\beta)\, d(x_{2n+1}, x_{2n+2})$

$\Rightarrow d(x_{2n}, x_{2n+1}) \geq \frac{(2\alpha+2\beta)}{1-(\alpha+\beta)} d(x_{2n+1}, x_{2n+2})$

$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq \frac{1-(\alpha+\beta)}{(2\alpha+2\beta)} d(x_{2n}, x_{2n+1})$

Therefore, $d(x_{2n+1}, x_{2n+2}) \leq hd(x_{2n}, x_{2n+1}), where\ h = \frac{1-(\alpha+\beta)}{(2\alpha+2\beta)}, 0 \leq h \leq 1.$

In general

$d(x_{2n}, x_{2n+1}) \leq hd(x_{2n-1}, x_{2n}) \leq \ldots\ldots\ldots\ldots h^{2n} d(x_0, x_1)$

So for $n < m$, we have

$d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n+2}) + \ldots\ldots\ldots + d(x_{2m-1}, x_{2m})$

$$\leq (h^{2n} + h^{2n+1} + \cdots\cdots\cdots\cdots + h^{2m-1})d(x_0, x_1)$$

$$\leq \frac{h^{2n}}{1-h} d(x_0, x_1)$$

Let $0 \leq c$ be given, choose a natural number N_1 such that $\leq \frac{h^{2n}}{1-h} d(x_0, x_1) \leq c$, for all $n \geq N_1$. Thus $d(x_{2n}, x_{2m}) \leq c$, for $n < m$.Therefore$\{x_{2n}\}$ is a Cauchy sequence in (X, d, k). Since (X, d, k) is a complete digital metric space, there exist $z \in X$ such that$x_{2n} \to z\ as\ n \to \infty$. If T_1 is continuous, then

$$d(T_1 z, z) \leq d(T_1x_{2n+1}, T_1 z) + d((T_1x_{2n+1}, z) \to 0 asn \to \infty.$$

$Since\ \ x_{2n} \to z\ and\ \ T_1x_{2n+1} \to T_1 x^*\ as\ \ n \to \infty$. Therefore $d(T_1 z, z)$ = 0. This implies that,$T_1 z$ = z. Hence z is a fixed point ofT_1

Similarly, it can be established that $T_2\, z$ = z. Therefore,$T_1\, z$ = z=,$T_2\, z$. Thus $z$is the common fixed point of pair of maps T_1 andT_2 . This completes the proof.

Theorem 3.5: Let (X, k) be a digital image where $X \subseteq Z^n$ and *k* is an adjacency relation in *X.* Let (X, d, k) be a complete digital metric space and suppose $T_1, T_2 : X \to X$ be any two continuous and onto mapping satisfying the condition

$$d(T_1x, T_2y) \geq \alpha \max\{d(x, y), d(x, T_1x), d(y, T_2y)\}$$

$$+\beta \max\{d(x, T_2y), d(x, y)\} + \gamma d(x, y)$$

for all *x,y*∈ X, where$\alpha \geq 0, \beta > 0, \gamma \leq 1$are constant, with $\alpha + \beta + \gamma > 1$. Then $T_1$and T_2have a common fixed point in *X.*

Proof: Let x_o be an arbitrary point in X. Since $T_1$and T_2be onto (surjective), there exist $x_o \in X$ and $x_1 \in X$

such that

$$T_1(x_1) = x_0, T_2(x_2) = x_1$$

In this way, we define the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ by

$x_{2n} = T_1 x_{2n+1}$for $n = 0,1,2,3$ ....and

$x_{2n+1} = T_2 x_{2n+2}$ for $n = 0,1,2,3$ … ….

Note that, If $x_{2n} = x_{2n+1}$ some n ≥ 1, then it is fixed point of T_1 and T_2.

Now putting $x = x_{2n+1}$and $y = x_{2n+2}$, we have

$d(x_{2n}, x_{2n+1}) = d(T_1 x_{2n+1}, T_2 x_{2n+2})$

$\Rightarrow d(x_{2n}, x_{2n+1}) \geq \alpha \max\{[d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, T_1 x_{2n+1}), d(x_{2n+2}, T_2 x_{2n+2})\}$

$+ \beta \max\{d(x_{2n+1}, T_2 x_{2n+2}), d(x_{2n+1}, x_{2n+2})\} + \gamma d(x_{2n+1}, x_{2n+2})$

$\geq \alpha \max\{[d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n}), d(x_{2n+2}, x_{2n+1})\}$

$+ \beta \max\{d(x_{2n+1}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} + \gamma d(x_{2n+1}, x_{2n+2})$

$\Rightarrow d(x_{2n}, x_{2n+1}) \geq (\alpha + \beta + \gamma)\, d(x_{2n+1}, x_{2n+2})$

$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq \frac{1}{(\alpha+\beta+\gamma)} d(x_{2n}, x_{2n+1})$

$\Rightarrow \quad d(x_{2n+1}, x_{2n+2}) \leq h d(x_{2n}, x_{2n+1}),\ where\ h = \frac{1}{(\alpha+\beta+\gamma)},\ 0 \leq h \leq 1$.

In general

$d(x_{2n}, x_{2n+1}) \leq h d(x_{2n-1}, x_{2n}) \leq \ldots, \ldots, \ldots . h^{2n} d(x_0, x_1)$

So for $n < m$, we have

$d(x_{2n}, x_{2m}) \leq d(x_{2n}, x_{2n+2}) + \ldots, \ldots, \ldots + d(x_{2m-1}, x_{2m})$

$\leq (h^{2n} + h^{2n+1} + \cdots\cdots\cdots\cdots + h^{2m-1}) d(x_0, x_1)$

$\leq \frac{h^{2n}}{1-h} d(x_0, x_1)$

Let $0 \leq c$ be given, choose a natural number N_1 such that $\frac{h^{2n}}{1-h} d(x_0, x_1) \leq c$, for all $n \geq N_1$. Thus $d(x_{2n}, x_{2m}) \leq c$, for $n < m$.Therefore$\{x_{2n}\}$ is a Cauchy sequence in (X, d, k). Since (X, d, k) is a complete digital metric space, there exits $z \in X$such that$x_{2n} \to z$ as $n \to \infty$. If T_1 is continuous, then

$$d(T_1 z, z) \leq d(T_1 x_{2n+1}, T_1 z) + d((T_1 x_{2n+1}, z) \to 0 asn \to \infty.$$

$Since\ x_{2n} \to z\ and\ T_1 x_{2n+1} \to T_1 z\ as\ n \to \infty$. Therefore $d(T_1 z, z) = 0$. This implies that, $T_1 x^* = x^*$. Hencex^* is a fixed point ofT_1

Similarly, it can be established that $T_2 z = z$ .Therefore, $T_1 z = z =, T_1 z$. Thus $z$is the common fixed point of pair of maps T_1 andT_2.This completes the proof.

CONCLUSION

Since a digital images be a combination of pixels in 2-D while voxels are in 3-D. And distance function between these pixels may be contractive or expansive. On this concept to find the diversified image we took two self-mapping over complete digital metric space via five various types of expansive conditions and proved fixed point theorems. We hope these theorems provide a new dimension of very traditional FPT to the modern theory of Image processing.

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