Bull. Cal. Math. Soc., **112,** (1) 31–36 (2020)

NEW GENERALIZED MEASURES OF ENTROPY

SANGEETA PANDEY[1], R.P. DUBEY[2] AND P. JHA[3]

(*Received 14 November 2019*)

Abstract. In the present paper, the measures of entropy are obtained by existing literatures. Due to the tremendous success of Shannons measure of entropy and the penchant of mathematicians for generalisation, a large number of efforts were made to generalise Shannons measure . In this paper ,some new measures of entropy and directed divergence have been generalised in different ways and this is done without introducing any additional parameters.We also studied their fundamental and desirable properties and presented these measures through graphs.

KEYWORDS : Entropy, directed divergence

1.1 Introduction. For a probability distribution $P = (p_1, p_2, \cdots, p_n)$. Shannon gave measure of entropy in 1948 as

$$S(P) = -\sum_{i=1}^{n} p_i \log p_i$$

In 1961, Renyi generalized it to give his measure of entropy of order α

$$\text{i.e., } R_\alpha(P) = \frac{1}{1-\alpha} \log \sum_{i=1}^{n} p_i^\alpha, \quad \alpha > 1, \alpha \neq 1$$

This was a parametric measure for each value of the parameter α. It reduces to the measure of entropy given by Shannon's measure by taking $\alpha \to 1$. After this, by introducing several parameters, different measures of entropy were introduced by different researchers.

In this paper we introduced new generalised measures of entropy without introducing any additional parameter.

1.2 Main results

1.2.1 Measure corresponding to Havrda and Charvat's Measure of entropy

$$H_1(P) = \frac{1}{1-\alpha} \left(\sum_{i=1}^{n} (p_i^\alpha + (1-p_i)^\alpha) - n \right), \quad \alpha \neq 1.$$

Properties :

1. $H_1(P)$ is a continuous function of p_i.

2. $H_1(P)$ is a permutationally symmetric function of p_i.

$$\text{i.e.,}\ \ H_1(p_i) = H_1(1-p_i)$$

3. $H_1(P)$ reduces to zero for degenerate distributions

$$\begin{aligned} D_1 &= (1,0,0,\cdots,0),\\ D_2 &= (0,1,0,\cdots 0),\cdots,\\ D_n &= (0,0,0,\cdots,1). \end{aligned}$$

4. When $p_1 = p_2 = p_3 = \cdots p_n = \frac{1}{n}$, then $H_1(P)$ has

$$\text{maximum value} = \frac{1}{1-\alpha}\left[n\left\{\frac{1+(n-1)^\alpha}{n^\alpha}\right\} - n\right]$$

5. concavity

$$\begin{aligned} \frac{\partial H_1(P)}{\partial P_i} &= \frac{\alpha}{1-\alpha}\left[p_i^{\alpha-1} - (1-p_i)^{\alpha-1}\right],\\ \frac{\partial^2 H_1(P)}{\partial P_i^2} &= -\alpha\left[p_i^{\alpha-2} + (1-p_i)^{\alpha-2}\right],\\ \frac{\partial^2 H_1(P)}{\partial P_i^2} &< 0, \quad \forall\, \alpha > 0,\ \alpha \neq 1. \end{aligned}$$

   So $H_1(P)$ is a concave function of $p - i$, $i = I$, to n

6. Monotonic nature

   When $\alpha = 0$, $H_1(P) = n$

   When $\alpha \to 1$, $\dfrac{dH_1(P)}{d\alpha} < 0 \quad \forall\, \alpha > 0,\ \alpha \neq 1$

When

$$\alpha \to \infty,\ H_1(P) \to 0$$

So, $H_1(P)$ is monotonic decreasing function of α

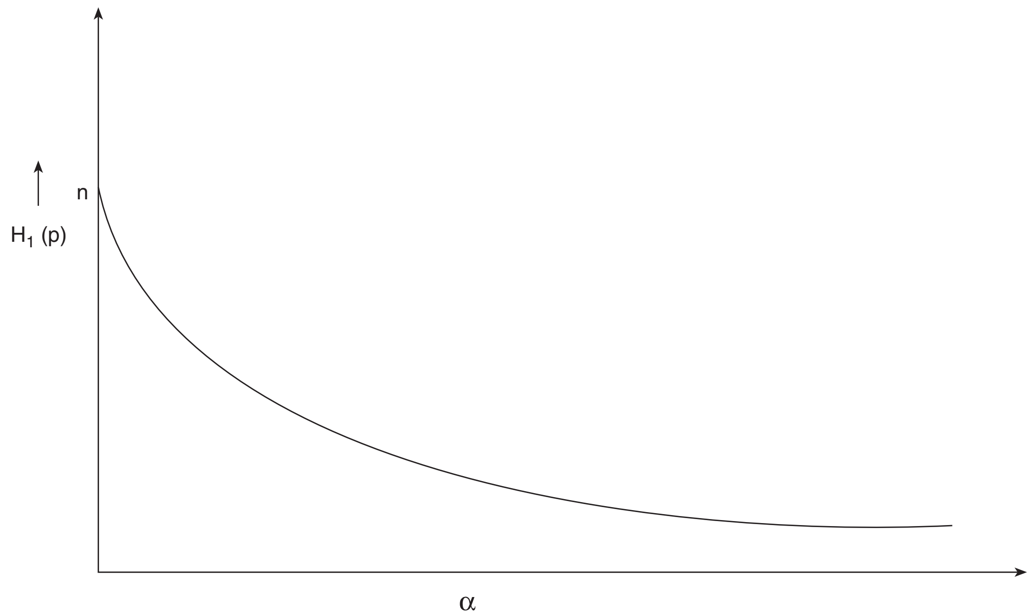

Fig. 1 : Monotonic behaviour of $H_1(P)$

7. Generalised measure of directed divergence corresponding to this entropy

$$D_1(P:Q) = \frac{1}{1-\alpha}\left[\sum \left\{p_i^{\alpha} q_i^{1-\alpha} + (1-p_i)^{\alpha}(1-q_i)^{1-\alpha}\right\} - n\right]$$

$$D_1(P:Q) > 0\ \ \forall\ \alpha > 0,\ \alpha \neq 1.$$

So directed divergence is monotonically increasing function of α.

1.2.2 Measure corresponding to Sharma and Taneja's measure of entropy without introducing new parameter

$$H_2(P) = \frac{1}{\beta-\alpha}\left(\sum_{i=1}^{n} (p_i^{\alpha} + (1-p_i)^{\alpha}) - \sum_{i=1}^{n}\left(p_i^{\beta} + (1-p_i)^{\beta}\right)\right),$$

Properties :

1. $H_2(P)$ is a continuous function of P_i.

2. $H_2(P)$ is a permutationally symmetric function of p_i.

$$\text{i.e., } H_2(p_i) = H_2(1 - p_i)$$

3. $H_2(P)$ reduces to zero for degenerate distributions are

$$\begin{aligned} D_1 &= (1, 0, 0, \cdots, 0), \\ D_2 &= (0, 1, 0, \cdots, 0), \cdots, \\ D_n &= (0, 0, 0, \cdots, 1). \end{aligned}$$

4. When $p - 1 = p_2 = p_1 = \cdots, p_n = \frac{1}{n}$, then $H_2(P)$ has

$$\text{Maximum value} = \frac{1}{\beta - \alpha}\left[n\left\{\frac{1 + (n-1)^\alpha}{n^\alpha} - \frac{1 + (n-1)^\beta}{n^\beta}\right\}\right]$$

5. For concavity

When $\alpha < 1$, $\beta \geq 1$, $\alpha \neq \beta$

$$\frac{\partial^2 H_2(P)}{\partial P_i^2} = \frac{1}{\beta - \alpha}\Big[\alpha(\alpha-1)p_i^{\alpha-2} + \alpha(\alpha-1)(1-P_i)^{\alpha-2} - \beta(\beta-1)p_i^{\beta-2}$$
$$\beta(\beta-1)(1-P_i)^{\beta-2}\Big] < 0$$

So $H_2(P)$ is concave for $\alpha < 1$, $\beta \geq 1$, $\alpha \neq \beta$

6. Monotonic behaviour

(i) when $\alpha = 0$ and $\beta = 1$, then $H_2(P) = n$

(ii) when $\alpha = 0$ and $\beta \to \infty$, then $H_2(P) = 0$

In this case function is monotonically decreasing.

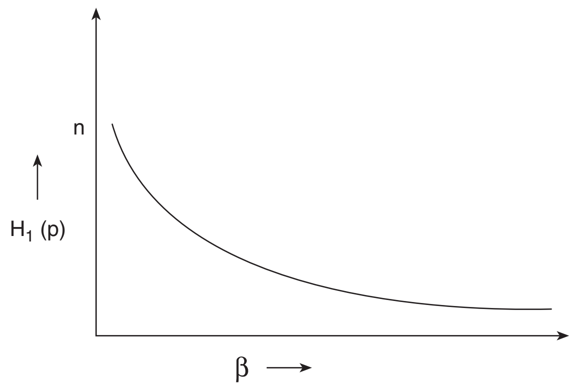

Fig. 2: Monotonic behaviour of $H_2(P)$

7. Generalised measure of Directed Divergence according to this measure of entropy is

$$D_2(P:Q) = \frac{1}{\beta-\alpha}\left[\sum_{i=1}^{n}\left(p_i^{\alpha}q_i^{1-\alpha}+(1-p_i)^{\alpha}(1-q_i)^{1-\alpha}\right) - \sum_{i=1}^{n}\left(p_i^{\beta}q_i^{1-\beta}+(1-p_i)^{\beta}(1-q_i)^{1-\beta}\right)\right]$$

which is monotonic increasing function.

1.3 Conclusion. Thus in this paper, new generalized measures of entropy and directed divergence have been introduced without any additional parameter. These generalizations of measures of entropy and measures of directed divergence satisfy all properties. These generalizations are important for developing measures of fuzzy entropy and measure of fuzzy directed divergence.

References

Havdra, J.H. and **Charvat, F.C.** (1967) : "Qualification Methods of Classification Processes of Concept of Structural a Entropy". *Kybernatics*, **3**, 30.

Kapur, J.N. (1967) : "Generalized Entropy of order α and β and Type β". *The Maths. Seminar* **4**, 78.

Kapur, J.N. (1986) : "Four Families of Measures of Entropy", *Ind. Jour. Pure and App. Maths.*, **17**(4), 429.

Kapur, J.N. (1996) : "Measure of Fuzzy Information." *Mathematical Sciences Trust Society*, New Delhi.

Pal, N.R. and **Pal, S.K.** (1999) : "Entropy: a new definitions and its applications." *IEEE Transaction on systems, Man and cybernetics*, **21**(5), 1260.

Renyi, A. (1961) : "On measures of entropy and information, Proceedings of the Fourth Berkeley symposium on Mathematical Statistics and probability." Berkeley; CA: University of California Press, 547.

Shannon, C.E. (1948) : "A Mathematical theory of communication." *Bell system Technical Journal*, **27**, 379, 623.

Sharma, B.D. and **Mittal, D.P.** (1975) : "New Non Additive Measure of Entropy for Discrete Probability Distribution", *J. Math. Sc.*, **10**, 28.

[1]DEPARTMENT OF MATHEMATICS
RAJIV GANDHI GOVT. P.G. COLLEGE
AMBIKAPUR (C.G.), INDIA

[2]DEPARTMENT OF MATHEMATICS
DR. C.V. RAMAN UNIVERSITY
KARGI ROAD, KOTA,
BILASPUR (C.G.), INDIA

[3]DEPARTMENT OF MATHEMATICS
GOVT. D.K.P.G. COLLEGE
BALODABAZAR (C.G.), INDIA