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Some Fixed Point Theorems for Contractive Mappings In Complex Valued Partial b-Metric Spaces

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ABSTRACT

The technique of fixed point is assumed a core and powerful tool in Nonlinear Functional Analysis because it plays an important role in pure and applied Mathematics. Fixed point theorem is the developments of new creative methods to use the fundamental tools of the functional analysis. Functional analysis embodies the abstract approach in analysis. It gives many fundamental notions relevant for the description, Analysis, Numerical Approximation, Medicinal Ecology, Agro industries and Computer Simulation process. It also discovers solutions to problems occurring in pure, applied and social sciences. In this way, Mathews introduced the notion of Partial Metric space as a part of the study of mentioned semantics of data flow networks related to Computer Science. There are many researchers extended the concept of Partial metric space such as Partial b-metric space, Complex Valued partial metric space, Fuzzy type Partial metric space and proved the existence of fixed point theorem via contraction mappings.

The aim of this research paper, to study of fixed point theorems for certain contractive mappings in Complex valued partial *b*-metric spaces. The present research work will attempt to extends, generalize and improve results propagated by mathematicians in this field.

KEYWORDS: Partial metric space, Partial b-metric space, fixed point, CPbMS

1. INTRODUCTION

Fixed point theorems give the condition under which maps have solutions. The theory itself is a beautiful mixture of Analysis, Topology and Geometry. Over the last years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of non –linear phenomena. However, for many practical situations the conditions that have to be imposed in order to guarantee the existence of fixed point are too strong. Considerable amount of researchers have been done on fixed point theorem for various type mapping in the last few years.

Fixed Point Theory Play's a major role in applications of many branches of Mathematics. In 1922, police mathematician Banach proved a very important result regarding a contraction mapping, known as the Banach contraction principle (Banach, 1922). Czerwik introduced the concept of b-metric space (Czerwik, 1993). Azam et al. gave the concept of complex valued metric space (Azam et al., 2011). Maheshwari et.al introduced to complex partial b-metric space and proved the existence of coupled fixed point result under contractive conditions in this space (Maheshwari et.al, 2021).

Wang et al. introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces (Wang et al., 1984). Mathews introduced the notion of partial metric space as a part of the study of mentioned semantics of data flow networks related to in computer science (Mathews, 1994).

2. PRELIMINARIES NOTES

First, we invite some standard notations and definitions of Complex valued b- metric space some Properties as follows :

An ordinary metric d is a real valued function from a set $X \times X$ into R where X is non empty set that is $d: X \times X \to R$. A complex number $z \in \mathbb{C}$ is an ordered pair of real numbers where first co-ordinate is called $Re(z)$ and second co-ordinate is called $Im(z)$. Thus a complex valued metric space d is a function from a set $X \times X$ into $\mathbb{C}$, where X is the non-empty set and $\mathbb{C}$ is the set of complex number .

That is $d: X \times X \to \mathbb{C}$ .Let $z_1, z_2 \in \mathbb{C}$ difine a partial order $\leq$ on $\mathbb{C}$ as follows

$z_1 \leq z_2$ iff $Re(z_1) \leq Re(z_2)$, $Im(z_1) \leq Im(z_2)$.

It follow that $z_1 \leq z_2$ if one of the following condition are satisfied:

$(i) Re(z_1) = Re(z_2) \quad and \ Im(z_1) < Im(z_2)$

$(ii) Re(z_1) < Re(z_2) \quad and \ Im(z_1) = Im(z_2)$

$(iii) Re(z_1) < Re(z_2) \quad and \ Im(z_1) < Im(z_2)$

$(iv) Re(z_1) = Re(z_2) \quad and \ Im(z_1) = Im(z_2)$

In (i),(ii),(iii) we have $|z_1| \leq |z_2|$. In (iv), we have $|z_1| = |z_2|$. So $|z_1| \leq |z_2|$ In particular $|z_1| \lneqq |z_2|$ if $z_1 \neq z_2$ and one of (i), (ii), (iii) is satisfy. In this case $|z_1| < |z_2|$. We will write $z_1 < z_2$ iff (iii) satisfy. Further

$$0 \leq z_1 \lneqq z_2 \Rightarrow |z_1| \leq |z_2|.$$

$$z_1 \leq z_2 \text{ and } z_2 < z_3 \Rightarrow z_1 < z_3.$$

Definition 2.1.(Azam et al., 2011): Let x be a nonempty set. Suppose that the mapping $d: X \times X \rightarrow \mathbb{C}$ satisfies the following conditions:

$(i) d(x,y) \geq 0 \text{ and } d(x,y) = 0 \Leftrightarrow x = y.$

$(ii) d(x,y) = d(y,x)$ $\qquad$ $(symmetic)$

$(iii) d(x,y) \leq d(x,z) + d(z,y)$ $\qquad$ $(the\ trinagle\ inequalities).$

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

Example 2.2. (Datta, et al., 2012): Let $X = \mathbb{C}$ Define the mapping $d: X \times X \rightarrow C$ by

$d(x,y) = i|x-y|$ $,\forall x,y \in X$. Then (X,d) is complex valued metric space.

Example 2.3. (Tanmoy, 2015): Let $X = \mathbb{C}$ Define the mapping $d: X \times X \rightarrow \mathbb{C}$ by

$$d(x,y) = e^{ik}|x-y| \text{ where } k \in R \text{ and}, \forall x,y \in X,$$

Where $k \in \left[0, \frac{\pi}{2}\right]$ $,\forall x,y \in X$ .Then (X,d) is called a complex valued metric space.

Definition 2.4.[36]: Let X be a nonempty set and mapping $d: X \times X \rightarrow \mathbb{C}$ satisfy the following conditions:

$(i) 0 \leq d(x,y) \text{ and } d(x,y) = 0 \text{ iff } x = y \quad \forall x,y \in X.$

$(ii)\ d(x,y) = d(y,x)$

$(iii) d(x,y) \leq S[d((x,z) + d(z,y)].$

Where $s \geq 1$ is a real number. Then d is called complex valued metric space and (X,d) is called complex valued b-metric space.

Example 2.5.(Rao, 2013): Let $X = [0,1]$. Define a complex valued metric $d: X \times X \rightarrow \mathbb{C}$ by $d(x,y) = |x-y|^2 + i|x-y|^2$ $,\forall x,y \in X$

Then (X,d) is a complex valued b-metric space with S=2.

Remark: If S=1, then the complex valued b-metric space always reduces to a complex valued metric space. Thus every complex valued metric space is a complex valued b-metric space, but not conversely. This generalizes the notation of a complex valued b-metric space over complex valued metric space.

Definition 2.6.(Rao, 2013): Let (X, d) be a complex valued b-metric space consider the following:

(i) A point $x \in X$ is called interior point of a set $A \subset X$ whenever point. There exists $0 < r \in C$ such that

$$B(x, r) = \{y \in X: d(x, y) < r\} \subseteq A.$$

(ii) A point $x \in X$ is called a limit point of a set A whenever there exists for every $0 < r \in C$,

$$B(x, r) \cap (A - X) = \emptyset.$$

(iii) A subset $A \subseteq X$ is called open whenever each element of A is an interior point of A.

(iv) A subset $A \subseteq X$ is called closed whenever each element of a A belong to A.

(v) A subbasis for a Housdroff topology τ on X is a family

$$F = \{B(x, r): x \in X \; and \; 0 < r\}.$$

Definition 2.7.(Rao, 2013): Let (X, d) be a complex valued b-metric space and $\{x_n\}$ a sequence in X and $x \in X$ consider the following:

(i) If for every $c \in C$,with $0 < r$,there is $n \in N$ such that for all $d(x_n, x) < c$. Then $\{x_n\}$ is said to be convergent , $\{x_n\}$ coverges to x and x is the limit point of $\{x_n\}$ .We donote this by $\lim_{n\to\infty} x_n = x \;\; or \; \{x_n\} \to x \; as \; n \to \infty$

(ii)If for every $c \in \mathbb{C}$ with $0 < r$, there is $n > N$ $d(x_n, x_{n+m}) < r$. where $m \in N$. Then $\{x_n\}$ is said to be Cauchy sequence.

(iii)If every Cuachy sequence in X is convergent, then (X, d) is said to be a complete complex valued b-metric space.

Lemma 2.8.(Rao, 2013)**:** let (X, d) be a complex valued b-metric space and Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and if only $|d(x_{n,} x|$ $\to 0$ as $n \to \infty$.

Lemma 2.9.(Rao, 2013)**:** let (X, d) be a complex valued b-metric space and Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and if only $|d(x_{n,} x_{n+m})| \to 0$ as $n \to \infty$,where $m \in N$.

Definition 2.10.(Matthews, 1994) **:** Let X be a non empty set. A partial metric space is a pair (X, d_p), where $d_p: X \times X \to R$ is such that

P_1: $0 \leq d_p(x,x) \leq d_p(x,y)$ (non-negativity and self-distances),

P_2: If $d_p(x,x) = d_p(x,y) = d_p(y,y)$, then $x = y$ (Indistancy implies equality),

P_3: $d_p(x,y) = d_p(y,x)$ (symmetry),

P_4: $d_p(x,z) \leq d_p(x,y) + d_p(y,z) - d_p(y,y)$ (triangularity),

Where d_p is called partial metric space.

It is clear that if $d_p(x,y) = 0$, then from (P_1) and (P_2), $x = y$. But if $x = y$, $d_p(x,y)$ may not be 0.

Definition 2.11. (Maheshwari et al., 2021)**:** A complex partial b-metric space on a non void set X is a function $d_{cpb} : X \times X \to C^+$ Such that for all $\lambda, \mu, \kappa \in X$:

(i) $d_{cpb}(\lambda,\mu) \leq d_{cpb}(\lambda,\mu) \leq 0$

(ii) $d_{cpb}(\lambda,\mu) = d_{cpb}(\mu,\lambda)$

(iii) $d_{cpb}(\lambda,\lambda) = d_{cpb}(\lambda,\mu) = d_{cpb}(\mu,\mu) \Leftrightarrow \lambda = \mu$

(iv) there exist a real number $s \geq 1$ such that

$$d_{cpb}(\lambda,\mu) \leq s[d_{cpb}(\lambda,\kappa) + d_{cpb}(\kappa,\mu)] - d_{cpb}(\kappa,\kappa).$$

A complex valued partial metric space is a pair of (X, d_{cpb}) such that X is a non empty set and is d_{cpb} complex partial b-metric space on X, the number s is called the coefficient of (X, d_{cpb}).

Definition 2.12.(Maheshwari et al., 2021)**:** Let (X, d_{cpb}) be a complex partial b-metric space with coefficient s. Let $< \lambda_n >$ be a sequence in X and $\lambda \in X$. Then

(i) The sequence $< \lambda_n >$ is said to be convergent with respect to d_{cpb} and converges to λ, if $\lim_{n\to\infty} d_{cpb}(\lambda_n, \lambda) = d_{cpb}(\lambda,\lambda)$.

(ii) The sequence $< \lambda_n >$ is said to be Cauchy sequence in (X, d_{cpb}), if $\lim_{n\to\infty} d_{cpb}(\lambda_n, \lambda_m)$ exists and finite.

(iii) (X, d_{cpb}) is said to be a complete complex partial b- metric space if every Cauchy sequence $< \lambda_n >$ in X there exists such that

$$\lim_{n\to\infty} d_{cpb}(\lambda_n, \lambda_m) = \lim_{n\to\infty} d_{cpb}(\lambda_n, \lambda) = d_{cpb}(\lambda,\lambda)$$

3. FIXED POINT THEOREMS FOR PAIR OF MAPPINGS IN COMPLEX PARTIAL b- METRIC SPACE

In this section, we explain the following fixed point theorem for a self mapping satisfying contractive condition in Complex valued b- Metric Spaces. Our results extend, generalize and improve the corresponding results.

Theorem 3.1. (Dubey et al., 2015)**:** Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $T: X \to X$ be a mapping satisfying the condition:

$$d(Tx,Ty) \leq \frac{\lambda\, d^2(x,y)}{1+d(x,y)} + \mu\, d(y,Ty)$$

for all $\forall x,y \in X$,where λ , μ are nonnegative reals with $s\lambda + \mu < 1$.Then T Has a unique fixed point in X.

Theorem 3.2. (Dubey et al., 2015)**:** Let (X,d) be a complete complex valued b-metric space with the coefficient $s \geq 1$ and $T_1, T_2: X \to X$ be a mapping satisfying

$$d(Tx,Ty) \leq \lambda d(x,y) + \frac{\mu d(x,Tx) d(y,Ty)}{d(x,Ty)+d(y,Tx)+d(x,y)},$$

for all $x,y \in X$ such that $x \neq y$, $d(x,Ty) + d(y,Tx) + d(x,y) \neq 0$, where λ, μ are nonnegative reals with $s\lambda + \mu < 1$ or $d(T_1x, T_2y) = 0$ if $d(x,Ty) + d(y,Tx) + d(x,y) = 0$. Then T Has a unique fixed point in X.

Now we have prove the following theorems 3.3 and 3.4 which is extension and generalization form of the above theorems 3.1 and 3.2 respectively.

Theorem 3.3: Let (X, d_{cpb}) be a complete complex valued partial b-metric space with the coefficient $s \geq 1$ and $T_1, T_2: X \to X$ be a mapping satisfying the condition:

$$d_{cpb}(T_1x, T_2y) \leq \frac{\lambda d_{cpb}{}^2(x,y)}{1+d_{cpb}(x,y)} + \mu d_{cpb}(y, T_2y) \qquad (1)$$

for all $\forall x,y \in X$,where λ , μ are nonnegative reals with $s\lambda + \mu < 1$.Then T_1 and T_2 have a unique common fixed point in X.

Proof. For any arbitrary point, $x_n \in X$. Define sequence $\{x_n\}$ in X such that

$x_{2n+1} = T_1 x_{2n}$ for $n = \{0,1,2,3,\ldots\}$

(2)

$x_{2n+2} = T_2 x_{2n+1}$ for $n = \{0,1,2,3,\dots\}$ (3)

Now, we show that the sequence $\{x_n\}$ is Cauchy : Let $x = x_{2n}$ & $y = x_{2n+1}$ in (1) we have

$$d_{cpb}(x_{2n+1}, x_{2n+2}) = d_{cpb}(T_1 x_{2n}, T_2 x_{2n+1})$$

$$\leq \frac{\lambda d_{cpb}{}^2(x_{2n}, x_{2n+1})}{1 + d_{cpb}(x_{2n}, x_{2n+1})} + \mu d_{cpb}(x_{2n+1}, T_2 x_{2n+1})$$

$$\leq \frac{\lambda d_{cpb}{}^2(x_{2n}, x_{2n+1})}{1 + d_{cpb}(x_{2n}, x_{2n+1})} + \mu d_{cpb}(x_{2n+1}, x_{2n+2}) \quad (4)$$

Which implies that

$$|d_{cpb}(x_{2n+1,} x_{2n+2})| \leq \lambda \frac{|d(x_{2n}, x_{2n+1})|}{|1 + d(x_{2n}, x_{2n+1})|} |d(x_{2n}, x_{2n+1})| + \mu |d(x_{2n+1}, x_{2n+2})| \quad (5)$$

Since $|1 + d_{cpb}(x_{2n}, x_{2n+1})| > |d_{cpb}(x_{2n}, x_{2n+1})|$, we get

$$|d_{cpb}(x_{2n+1}, x_{2n+2})| \leq \lambda |d_{cpb}(x_{2n}, x_{2n+1})| + \mu |d_{cpb}(x_{2n+1}, x_{2n+2})| \quad (6)$$

and hence

$$|d_{cpb}(x_{2n+1}, x_{2n+2})| \leq \frac{\lambda}{1-\mu} |d_{cpb}(x_{2n}, x_{2n+1})| \quad (7)$$

Similarly, we obtain

$$|d_{cpb}(x_{2n+2}, x_{2n+3})| \leq \frac{\lambda}{1-\mu} |d_{cpb}(x_{2n+1}, x_{2n+2})| \quad (8)$$

Since $s\lambda + \mu < 1$ and $s \geq 1$, we get $\lambda + \mu < 1$.

Therefore, with $\delta = \frac{\lambda}{1-\mu} < 1$ and for all $n \geq 0$. and consequently, we have

$$|d_{cpb}(x_{2n+1}, x_{2n+2})| \leq \delta |d_{cpb}(x_{2n}, x_{2n+1})| \leq \delta^2 |d_{cpb}(x_{2n-1}, x_{2n})| \leq \dots \leq \delta^{2n+1} |d_{cpb}(x_0, x_1)| \quad (9)$$

Thus for any $m > n, m, n \in \mathbb{N}$ and since $s\delta = \frac{s\lambda}{1-\mu} < 1$, we get

$$|d_{cpb}(x_{2n}, x_{2m})| \leq s |d_{cpb}(x_{2n}, x_{2n+1})| + s |d_{cpb}(x_{2n+1}, x_{2m})|$$

$$\leq s |d_{cpb}(x_{2n}, x_{2n+1})| + s^2 |d_{cpb}(x_{2n+1}, x_{2n+2})| + s^2 |d_{cpb}(x_{2n+2}, x_{2m})|$$

$$\leq s |d_{cpb}(x_{2n}, x_{2n+1})| + s^2 |d_{cpb}(x_{2n+1}, x_{2n+2})| + s^3 |d_{cpb}(x_{2n+2}, x_{2n+3})| + s^3 |d_{cpb}(x_{2n+3}, x_{2m})|$$

$\dots, \dots, \dots,$

$$\left|d_{cpb}(x_{2n},x_{2m})\right| \le s\left|d_{cpb}(x_{2n},x_{2n+1})\right| + s^2\left|d_{cpb}(x_{2n+1},x_{2n+2})\right| + s^3\left|d_{cpb}(x_{2n+2},x_{2n+3})\right| + \dots$$

$$+ s^{2m-2n-1}\left|d_{cpb}(x_{2m-2},x_{2m-1})\right| + s^{2m-2n}\left|d_{cpb}(x_{2m-1},x_{2m})\right| \quad (10)$$

By using (9), we get

$$\left|d_{cpb}(x_{2n},x_{2m})\right| \le s\delta^{2n}\left|d_{cpb}(x_0,x_1)\right| + s^2\delta^{2n+1}\left|d_{cpb}(x_0,x_1)\right| + s^3\delta^{2n+2}\left|d_{cpb}(x_0,x_1)\right| + \dots$$

$$+ s^{2m-2n-1}\delta^{2m-2}\left|d_{cpb}(x_0,x_1)\right| + s^{2m-2n}\delta^{2m-1}\left|d_{cpb}(x_{2m-1},x_{2m})\right|$$

$$= \sum_{i=1}^{2m-2n} s^i\delta^{i+2n-1}\left|d_{cpb}(x_0,x_1)\right| \quad (11)$$

$$\le \sum_{i=1}^{\infty} s^t\delta^{t+2n-1}\left|d_{cpb}(x_0,x_1)\right|$$

$$= \frac{s\delta^{2n}}{1-s\delta}\left|d_{cpb}(x_0,x_1)\right| \to 0\, as\ n,m \to \infty \quad (12)$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_n \to u$ as $n \to \infty$. Suppose that is not possible; then there exists $z \in X$ such that

$$\left|d_{cpb}(u,T_1u)\right| = |z| > 0 \quad (13)$$

Now,

$$z = \left|d_{cpb}(u,T_1u)\right| \le sd_{cpb}(u,x_{2n+2}) + sd_{cpb}(x_{2n+2},T_1u)$$

$$= sd_{cpb}(u,x_{2n+2}) + sd_{cpb}(T_2x_{2n+2},T_1u) \quad (14)$$

$$\le sd_{cpb}(u,x_{2n+2}) + \frac{s\lambda d_{cpb}{}^2(x_{2n+1},u)}{1+d_{cpb}(x_{2n+1},u)} + s\mu d_{cpb}(u,T_1u)$$

which implies that

$$|z| = \left|d_{cpb}(u,T_1u)\right| \le s\left|d_{cpb}(u,x_{2n+2})\right| + \frac{s\lambda\left|d_{cpb}{}^2(x_{2n+1},u)\right|}{1+\left|d_{cpb}(x_{2n+1},u)\right|} + s\mu\left|d_{cpb}(u,T_1u)\right|$$

(15)

Taking the limit of (16) as $n \to \infty$, we obtain that $|z| = \left|d_{cpb}(u,T_1u)\right| \le 0$, a contradiction with (13).

So $|z| = 0$. Hence $T_1u = u$.

Similarly, we can show that $T_1u = u$.

Now show that T_1 and T_2 have unique common fixed point of T_1 and T_2. To show this, assume that u^* is another fixed point of T_1 and T_2 . Then,

$$d_{cpb}(u,u^*) = d_{cpb}(T_1u,T_2u^*) \le \frac{\lambda d_{cpb}(u,u^*)}{1+d_{cpb}(u,u^*)} + \mu d_{cpb}(u^*,T_2u^*)$$

(16)

So

$$\left|d_{cpb}(u,u^*)\right| \le \lambda \frac{\left|d_{cpb}{}^2(u,u^*)\right|}{\left|1+d_{cpb}(u,u^*)\right|} + \mu\left|d_{cpb}(u^*,T_2u^*)\right|$$

(17)

Since

$$\left|1+d_{cpb}(u,u^*)\right| > \left|d_{cpb}(u,u^*)\right|$$

(18)

Therefore

$$\left|d_{cpb}(u,u^*)\right| < \lambda\left|d_{cpb}(u,u^*)\right| + \mu\left|d_{cpb}(u,u^*)\right|$$

(19)

$$= \lambda\left|d_{cpb}(u,u^*)\right|,$$ a contradiction.

So, $u = u^*$, which proves the uniqueness of fixed point in X. This completes the proof.

Theorem 3.3.4: Let (X, d_{cpb}) be a complete complex valued partial b-metric space with the coefficient $s \ge 1$ and $T_1, T_2: X \to X$ be a mapping satisfying

$$d_{cpb}(T_1x,T_2y) \le \lambda d_{cpb}(x,y) + \frac{\mu d_{cpb}(x,Tx_1)d_{cpb}(y,T_2y)}{d_{cpb}(x,T_2y)+d_{cpb}(y,T_1x)+d_{cpb}(x,y)} \qquad \ldots\ldots\ldots\ldots \ (20)$$

for all $x, y \in X$ such that $x \ne y$, $d_{cpb}(x,T_2y)+d_{cpb}(y,T_1x)+d_{cpb}(x,y) \ne 0$, where λ, μ are nonnegative real numbers with $s\lambda + \mu < 1$ or $d_{cpb}(T_1x,T_2y)$ if

$d_{cpb}(x,T_2y)+d_{cpb}(y,T_1x)+d_{cpb}(x,y) = 0$.

Then T_1 & T_2 have a unique common fixed point in X.

Proof. For any arbitrary point, $x_0 \in X$.Define sequence $\{x_n\}$ in X such that

$$x_{2n+1} = T_1x_{2n} \quad \text{for} \ \ n = (0,1,2,3 \ldots.) \qquad (21)$$

$$x_{2n+2} = T_2x_{2n+1} \quad \text{for} \ \ n = (0,1,2,3 \ldots.) \qquad (22)$$

Now, we show that the sequence $\{x_n\}$ is Cauchy: Let $x = x_{2n}$ & $y = x_{2n+1}$ in (20) we have

$$d_{cpb}(x_{2n+1}, x_{2n+2}) = d_{cpb}(Tx_{2n}, T_2x_{2n+1})$$
$$\leq \lambda d_{cpb}(x_{2n}, x_{2n+1}) + \frac{\mu d_{cpb}(x_{2n}, Tx_{2n}) d_{cpb}(x_{2n+1}, T_2x_{2n+1})}{d_{cpb}(x_{2n}, T_2x_{2n+1}) + d_{cpb}(x_{2n+1}, T_1x_{2n}) + d_{cpb}(x_{2n}, x_{2n+1})}$$
$$\leq \lambda d_{cpb}(x_{2n}, x_{2n+1}) + \frac{\mu d_{cpb}(x_{2n}, x_{2n+1}) d_{cpb}(x_{2n+1}, x_{2n+2})}{d_{cpb}(x_{2n}, x_{2n+2}) + d_{cpb}(x_{2n+1}, x_{2n+1}) + d_{cpb}(x_{2n}, x_{2n+1})}$$

(23)

which implies that

$$|d_{cpb}(x_{2n+1}, x_{2n+2})| \leq \lambda |d_{cpb}(x_{2n}, x_{2n+1})| + \frac{\mu |d_{cpb}(x_{2n+1}, x_{2n+2})|}{|d_{cpb}(x_{2n}, x_{2n+2})| + |d_{cpb}(x_{2n}, x_{2n+1})|} |d_{cpb}(x_{2n}, x_{2n+1})|$$

(24)

Since

$$|d_{cpb}(x_{2n+1}, x_{2n+2})| \leq |d_{cpb}(x_{2n+1}, x_{2n})| + |d_{cpb}(x_{2n}, x_{2n+2})| - |d_{cpb}(x_{2n}, x_{2n})|$$
$$|d_{cpb}(x_{2n+1}, x_{2n+2})| \leq |d_{cpb}(x_{2n+1}, x_{2n})| + |d_{cpb}(x_{2n}, x_{2n+2})|$$

(25)

Therefore

$$|d_{cpb}(x_{2n+1}, x_{2n+2})| \leq \lambda |d_{cpb}(x_{2n}, x_{2n+1})| + \mu |d_{cpb}(x_{2n}, x_{2n+1})|$$
$$\leq (\lambda + \mu) |d_{cpb}(x_{2n}, x_{2n+1})|$$

(26)

Similarly, we obtain

$$|d_{cpb}(x_{2n+2}, x_{2n+3})| \leq (\lambda + \mu) |d_{cpb}(x_{2n+1}, x_{2n+2})|$$

(27)

Since $s\lambda + \mu < 1$ and $s \geq 1$, we get $\lambda + \mu < 1$.

Therefore, with $\delta = \lambda + \mu < 1$ and for all $n \geq 0$ and consequently, we have

$$|d_{cpb}(x_{2n+1}, x_{2n+2})| \leq \delta |d_{cpb}(x_{2n}, x_{2n+1})| \leq \delta^2 |d_{cpb}(x_{2n-1}, x_{2n})| \leq \ldots \leq \delta^{2n+1} |d_{cpb}(x_0, x_1)|$$

(28)

Thus, for any $m > n$, $m, n \in \mathbb{N}$, we have

$$|d_{cpb}(x_{2n}, x_{2m})| \leq s |d_{cpb}(x_{2n}, x_{2n+1})| + s |d_{cpb}(x_{2n+1}, x_{2m})| - s |d_{cpb}(x_{2n+1}, x_{2n+1})|$$

$$\leq s\left|d_{cpb}(x_{2n},x_{2n+1})\right| + s\left|d_{cpb}(x_{2n+1},x_{2m})\right|$$

$$\leq s\left|d_{cpb}(x_{2n},x_{2n+1})\right| + s^2\left|d_{cpb}(x_{2n+1},x_{2n+2})\right| + s^2\left|d_{cpb}(x_{2n+2},x_{2m})\right|$$

$$\leq s\left|d_{cpb}(x_{2n},x_{2n+1})\right| + s^2\left|d_{cpb}(x_{2n+1},x_{2n+2})\right| + s^3\left|d_{cpb}(x_{2n+2},x_{2n+3})\right| + s^3\left|d_{cpb}(x_{2n+2},x_{2m})\right|$$

(29)

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$$\left|d_{cpb}(x_{2n},x_{2m})\right| \leq s\left|d_{cpb}(x_{2n},x_{2n+1})\right| + s^2\left|d_{cpb}(x_{2n+1},x_{2n+2})\right| + s^3\left|d_{cpb}(x_{2n+2},x_{2n+3})\right| + \dots$$

$$\leq s^{2m-2n-1}\left|d_{cpb}(x_{2m-2},x_{2m-1})\right| + s^{2m-2n}\left|d_{cpb}(x_{2m-1},x_{2m})\right|$$

By using (29), we get

$$\left|d_{cpb}(x_{2n},x_{2m})\right| \leq s\delta^{2n}\left|d_{cpb}(x_0,x_1)\right| + s^2\delta^{2n+1}\left|d_{cpb}(x_0,x_1)\right| + s^3\delta^{2n+2}\left|d_{cpb}(x_0,x_1)\right| + \dots$$

$$+ s^{2m-2n-1}\delta^{2m-2}\left|d_{cpb}(x_0,x_1)\right| + s^{2m-2n}\delta^{2m-1}\left|d_{cpb}(x_0,x_1)\right|$$

$$= \sum_{i=1}^{2m-2n} s^i\delta^{i+2n-1}\left|d_{cpb}(x_0,x_1)\right|$$

(30)

$$\leq \sum_{i=1}^{\infty} s^t\delta^{t+2n-1}\left|d_{cpb}(x_0,x_1)\right|$$

$$= \frac{s\delta^{2n}}{1-s\delta}\left|d_{cpb}(x_0,x_1)\right| \to 0\, as\ n,m \to \infty \tag{31}$$

Thus, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists some $u \in X$ such that $x_{2n} \to u$ as $n \to \infty$. Suppose that is not possible; then there exists $z \in X$ such that

$$\left|d_{cpb}(u,T_1u)\right| = |z| > 0$$

(32)

So by using the triangular inequality and (20), we get

$$z = d_{cpb}(u,T_1u)$$

$$\leq sd_{cpb}(u,x_{2n+2}) + sd_{cpb}(x_{2n+2},T_1u) - d_{cpb}(x_{2n+2},x_{2n+2})$$

$$= sd_{cpb}(u,x_{2n+2}) + sd_{cpb}(x_{2n+2},T_1u)$$

$$\leq sd_{cpb}(u,x_{2n+2}) + sd_{cpb}(T_2x_{2n+1},T_1u)$$

$$\leq sd_{cpb}(u,x_{2n+2})+s\lambda d_{cpb}(T_2x_{2n+1},u)+\frac{s\mu d_{cpb}(x_{2n+1},T_1x_{2n+1})d_{cpb}(u,T_2u)}{d_{cpb}(x_{2n+1},T_2u)+d_{cpb}(u,T_1x_{2n+1})+d_{cpb}(x_{2n+1},u)}$$

(33)

$$\leq sd_{cpb}(u,x_{2n+2})+s\lambda d_{cpb}(T_2x_{2n+1},u)+\frac{s\mu d_{cpb}(x_{2n+1},T_1x_{2n+1})d_{cpb}(u,T_2u)}{d_{cpb}(x_{2n+1},T_2u)+d_{cpb}(u,T_1x_{2n+1})+d_{cpb}(x_{2n+1},u)}$$

$$\leq sd_{cpb}(u,x_{2n+2})+s\lambda d_{cpb}(x_{2n+1},u)+\frac{s\mu d_{cpb}(x_{2n+1},x_{2n+2})d_{cpb}(u,T_2u)}{d_{cpb}(x_{2n+1},T_2u)+d_{cpb}(u,x_{2n+2})+d_{cpb}(x_{2n+1},u)}$$

which implies that

$$|z|=|d_{cpb}(u,T_1u)|$$

$$\leq s|d_{cpb}(u,x_{2n+2})|+s\lambda|d_{cpb}(x_{2n+1},u)|+\frac{s\mu|d_{cpb}(x_{2n+1},x_{2n+2})||d_{cpb}(u,T_1u)|}{d_{cpb}(x_{2n+1},T_1u)+d_{cpb}(u,x_{2n+2})+d_{cpb}(x_{2n+1},u)}$$

(34)

Taking the limit of (34) as $n\to\infty$, we obtain that $|z|=|d_{cpb}(u,T_1u)|\leq 0$, a contradiction with (32).

So $|z|=0$. Hence $T_1u=u$.

Similarly $T_2u=u$.

Now show that T_1 and T_2 have unique common fixed point of T_1 and T_2.To show that u^* is another fixed point of T_1 and T_2 .Then,

$$d_{cpb}(u,u^*)=d_{cpb}(T_1u,T_2u)$$

$$\leq \lambda d_{cpb}(u,u^*)+\frac{\mu d_{cpb}(u,T_1u)d_{cpb}(u^*,T_2u^*)}{d_{cpb}(u,T_2u^*)+d_{cpb}(u^*,T_1u)+d_{cpb}(u,u^*)}$$

(35)

so that

$$|d_{cpb}(u,u^*)|\leq \lambda|d_{cpb}(u,u^*)|+\frac{\mu|d_{cpb}(u,T_1u)||d_{cpb}(u^*,T_2u^*)|}{|d_{cpb}(u,T_2u^*)|+|d_{cpb}(u^*,T_1u)|+|d_{cpb}(u,u^*)|} \quad (36)$$

$$<\lambda|d_{cpb}(u,u^*)|, \text{ a contradiction.}$$

So $u=u^*$, which proves the uniqueness of fixed point in X. This complete the proof

Now, we consider the following case:

$d_{cpb}(x_{2n},T_2x_{2n+1})+d_{cpb}(x_{2n+1},T_1x_{2n})+d_{cpb}(x_{2n},x_{2n+1})=0$ (for any n) implies that $d_{cpb}(T_1x_{2n},T_2x_{2n+1})=0$ so that $x_{2n}=T_1\,x_{2n}=x_{2n+1}=T_2\,x_{2n+1}=x_{2n+2}$. Thus we have

$x_{2n+1} = T_1\, x_{2n} = x_{2n}$, so there exists $K_1$and l_1 such that $K_1 = T_1 l_1 = l_1$. Using foregoing arguments, one can also show that there exists K_2 and l_2 such that $K_2 = T_2 l_2 = l_2$.

As $d_{cpb}(l_1, T_2 l_2) + d_{cpb}(l_2, T_1 l_1) + d_{cpb}(l_1, l_2) = 0$ (due to definition) implies

$d_{cpb}(T_1 l_1, T_2 l_2) = 0$, $K_1 = T_1\, l_1 = T_2 l_2 = K_2$, which in turn yields that $K_1 = T_1\, l_1 = T_1\, K_1$. Similarly, one can also have $K_2 = T_2\, K_2$. As $K_1 = K_2$ implies $T_1\, K_1 = K_1$, therefore $K_1 = K_2$ is fixed point of T. We now prove that T_1 and T_2 have unique common fixed point. For this, assume that K_1^*in X is another fixed point of T. Then we have $T_2 K_1^* = K_1^*$.

As $d_{cpb}(K_1, T_2 K_1^*) + d_{cpb}(K_1^*, T_1 K_1) + d_{cpb}(K_1, K_1^*) = 0$,

Therefore, $d_{cpb}(K_1, K_1^*) + d_{cpb}(T_1 K_1, T_2 K_1^*) = 0$. This implies that $K_1 = K_1^*$ which proves the uniqueness of common fixed point in$X$. This completes the proof of the theorem.

CONCLUSION

In this research paper, we discussed about proposed system of fixed point theory in Partial metric space, complex valued b- metric space and Complex valued partial b- metric space and established some fixed point theorems for contractive mapping in Complex valued partial b- metric space. The present work has been improved and extended for mappings which satisfy contractive conditions in a setting of Complex valued Partial b-metric spaces. Further work will be for large amount data set having for three, four and six mappings satisfying generalized contractive type condition in a complete Partial cone metric, Complex Partial metric space, Complex Partial b-metric space and other generalize form of metric space with real world applications.

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